Arithmetic universes: Home of free algebras

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"Partial Horn logic and cartesian categories"

Palmgren and Vickers 2007 [PV07]

Background: Initial Model Theorem (IMT) (see [BW84])

Every cartesian theory has an initial model
Cartesian = essentially algebraic = finite limit theory
Hence also free algebras, generators and relations.

[PV07] Simplify using logic of partial terms

Theories simplified by Horn clause axioms in a partial logic of $=$ and $\wedge$.

Proof of IMT uses simple "term/congruence" construction from algebraic case

Example: Classifying categories as initial models.

Outline

Part I: AUs home of free algebras

Part II: Algebraic approach to classifying categories

Part III: Generalized topological spaces

AUs as foundations for continuous mathematics.

Pt I: AUs home of free algebras

Arithmetic universe (AU) = pretopos with parametrized list objects [Mai10]

They hit sweet spot of Initial Model Theorem:

- Theory of AUs is cartesian.
- Internal logic of AUs supports IMT [Mai05, Mai06].
  – and even term/congruence construction.

More general than elementary toposes with nno.

History of AUs

Early work (Joyal, Wraith) largely unpublished – apologies for any misrepresentations!

Joyal – used initial AU for account of Gödel's Incompleteness Theorem. (See [vG20].)

Wraith – reported it at 1985 "Categories in Computer Science" – but didn't write it up.

Both were aware of potential for internal free algebras.

Vickers [Vic99] suggested AUs for base-independent geometric reasoning.

Maietti (eg [Mai10, Mai06]) was the first to set out the current definition, and proved its major properties.

Maietti, Vickers, Hazratpour [MV12, Vic19, Vic17, HV20] develop ideas of AUs as generalized spaces.

Taylor [Tay05] – category of overt discrete spaces in ASD is an AU.

Initial model theorem – simple case

Algebraic theory = finitary operators + equational axioms

Theorem

Every algebraic theory has an initial model.

Proof.

"Term/congruence" construction.

1. Form term algebra to interpret signature (operators).
2. Generate congruence inductively from axioms.
3. Factor out congruence to model axioms.

□

Initial models ⇒ free algebras, and generators and relations

$\mathbb{T}$-model $\mathbb{T}\langle G \mid R \rangle$ presented by generators and relations

Take theory for $\mathbb{T}$

$+$ constants for generators in G

$+$ axioms for relations in R.

Model of this $=$ $\mathbb{T}$-model equipped with a function from G that respects relations in R.

$\mathbb{T}\langle G \mid R \rangle$ is the initial such.

Free models are for the case $R = \emptyset$.

From algebraic to essentially algebraic (cartesian)

The following are equivalent to each other, and to *finite limit sketches*. (Overview in [PV07].)

Essentially algebraic theories

- Finitary *partial* operators, each with domain of definition defined using equations involving previous operators.
- Axioms $s = t$ whenever both sides defined.

Cartesian theories [Joh02, D1.3.4]

- *Regular* first order theories (logic of $=, \wedge, \exists$).
- Axioms are *sequents* $\phi \vdash^{\vec{x}} \psi$ in *context* $\vec{x}$ listing available free variables.
- Each $\exists$ in an axiom must be for *unique* existence, provably from previous axioms.

Every cartesian theory has an initial model [BW84].

Examples of cartesian theories

Categories

Two sorts: objects and arrows
Composition is partial binary operator on arrows. Definedness, composability, given by an equation.

Categories + structure

eg pullback cones, given as partial binary operators on arrows forming cospan.
eg enrichment as operators on hom-sets.

eg elementary toposes with nno

Underlies methods of Lambek and Scott [LS86]

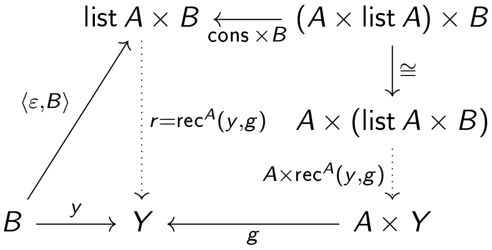
eg arithmetic universes (AUs)

Example of cartesian theory: Arithmetic universes (AUs)

= pretoposes with parametrized list objects [Mai10]

Parametrized list object list A

list A	type of finite lists of elements of A
$\varepsilon\colon 1 \to \text{list } A$	empty list []
$\text{cons}\colon A \times \text{list } A \to \text{list } A$	$a : l$ is l with a appended at front.



$$r([], b) = y(b)$$
$$r(a : x, b) = g(a, r(x, b))$$

nno $\mathbb{N} = \text{list } 1$

Example of cartesian theory: Arithmetic universes (AUs)

Pretoposes

Finite limits, finite coproducts, coequalizers of equivalence relations.
Axioms to make them cooperate.
In presence of list objects, they have transitive closures of binary relations and (hence) all finite coequalizers.

Theory of AUs is cartesian

Initial models: every cartesian theory has one [BW84]

BUT … term/congruence construction has problems.

Take essentially algebraic theory, with partial operators.

1. Form term algebra to interpret signature (operators).
2. Generate congruence inductively from axioms.
3. Factor out congruence to model axioms.

(1) – want *defined* terms.

(3) – creates more equations, hence more defined terms.

Iterate???

Initial models: every cartesian theory has one [PV07]

Use logic of *partial* terms

- Existence is self-equality.
- Straightforward adaptation of first-order logic as presented in [Joh02, D1.3]

Term/congruence using partial terms.

1. Form term algebra to interpret signature (operators).
2. Generate congruence inductively from axioms.
3. Factor out congruence to model axioms.

(1) – use *partial* terms.
(2) – generate *partial* congruence, not necessarily reflexive.
(3) – factor out partial congruence, ie congruence on self-equal terms.

Quasi-equational theories [PV07]

Quasi-equational theory is *Horn* theory.

- Signature is sorts S and operators O – no relation symbols.
- Logical connectives are $=, \wedge$
- Axioms are sequents in context.

$$\text{conjunction of equations} \vdash^{\vec{y}} \text{conjunction of equations}$$

They are equivalent to cartesian theories.

Term/congruence proof of IMT [PV07].

1. Express cartesian theory in quasi-equational form.
2. Use term/congruence construction for partial terms.

AUs support –

- Usual list operations for list A –
- – including concatenation $+\!\!+$, making list A free monoid on A.
- Arithmetic on $\mathbb{N} = \text{list}\,1$.
- [Mai10] Free categories on graphs, free category action from graph actions.

Initial models

- Maietti [Mai05, Mai06] using type theory
- Can also replicate term/congruence construction.
  Use reverse Polish notation to represent partial terms and proof terms.

AUs as "sweet spot" for IMT

IMT both – valid within AUs
– and can be used to present them.

Internalization

Special case:

The initial AU has an internal initial AU.
Original motivation.
Joyal (unpublished; but see [vG20]): explicit concrete construction.
Existence of $\mathbb{N}$ gives arithmetic.
Exhibits Gödel incompleteness – external = truth, internal = proof.

More generally – nested internalization

Constructions in AU logic can be carried out at different levels.
Mathematical consequences? Still little understood.

Pt II: Algebraic approach to classifying (syntactic) categories

Let $\mathcal{L}$ (for *logic*) be a cartesian theory of categories+structure

Algebra	**Logic**
Presentation $\mathbb{T}$	"$\mathcal{L}$-theory"
Generators	Signature (sorts and symbols)
Relations	Axioms

Correspondence fuzzy!

eg relations can say some sorts *derived* from other ingredients.

Assumption: presentation more important than separation signature/axioms. – cf. sketches

eg $\mathcal{L}$ = **AU**

Sorts can be derived as limits, colimits, list objects, and more general free constructions.

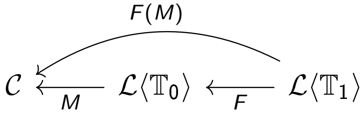
$\mathcal{L}\langle\mathbb{T}\rangle$ = classifying category for $\mathbb{T}$ (wrt $\mathcal{L}$)

$\mathcal{L}$-functors $(\mathcal{L}\langle\mathbb{T}\rangle \to \mathcal{C})$ *correspond to* $\mathbb{T}$-models in $\mathcal{C}$

Classifying category as "class of $\mathbb{T}$-models"

$\mathcal{L}$-functors $F\colon \mathcal{L}\langle\mathbb{T}_0\rangle \leftarrow \mathcal{L}\langle\mathbb{T}_1\rangle$ as model transformers

Composition with F maps $\mathbb{T}_0$-models into $\mathbb{T}_1$-models (in any $\mathcal{C}$).



How is F defined? – $\mathbb{T}_1$-model in $\mathcal{L}\langle\mathbb{T}_0\rangle$

Let M_0 be a model of $\mathbb{T}_0$. … various $\mathcal{L}$-constructions … … finish with $\mathbb{T}_1$-model. Call it $F(M_0)$.

$F(M)$ is substitution $F(M/M_0)$.

Formal parameter M_0 is generic (walking) model in $\mathcal{L}\langle\mathbb{T}_0\rangle$.
– $\mathcal{L}$-constructions all in $\mathcal{L}\langle\mathbb{T}_0\rangle$

Questions

Can this be made to work for type theory?

Does $\mathbf{AU}\langle\mathbb{T}\rangle$ classify models in ambient AU?

Each $\mathcal{C}$ is *small* – internal in ambient logic.
Can a model with sorts etc. *indexed by* $\mathbb{T}$ be extended to indexation by $\mathbf{AU}\langle\mathbb{T}\rangle$?

Strictness

Problem – see [MV12]

Strictness

$\mathcal{L}\langle\mathbb{T}\rangle$ = classifying category for $\mathbb{T}$ (wrt $\mathcal{L}$)

$\mathcal{L}$-functors ($\mathcal{L}\langle\mathbb{T}\rangle \to \mathcal{C}$) *correspond to* $\mathbb{T}$-models in $\mathcal{C}$
Strict $\mathcal{L}$-functors? Strict $\mathbb{T}$-models?

"Corresponds" – up to isomorphism or (usual interpretation) equivalence?

For algebra, syntax: require up to iso, and everything strict

For semantics: require non-strict, so up to equivalence.

Using *sketches* to handle both strict and non-strict [Vic19]

Restricted sketches, "contexts", have
every non-strict model has a canonical strict isomorph.
Then $\mathcal{L}\langle\mathbb{T}\rangle$ also classifies non-strict models (up to equivalance).
2-cat $\mathfrak{Con}$ of AU-contexts as generalized spaces.

Pt III: Generalized topological spaces

Classifying toposes are same idea

– but complicated by need for infinite disjunctions.

Classifying *topos* as "(generalized) *space* of $\mathbb{T}$-models"

$\mathcal{L}$: Logic $\mathcal{L}$ needs arbitrary disjunctions, to match unions of opens – $\mathcal{L}$ = **geometric** logic (Need arbitrary coproducts too.)

$\mathbb{T}$: **Point-free** topology. $\mathbb{T}$ = geometric theory of points.

$\mathcal{S}$: "Arbitrary" = $\mathcal{S}$-indexed, $\mathcal{S}$ = your favourite elementary topos + nno "of sets".

$\mathcal{S}[\mathbb{T}]$: Classifying *topos*.

point-set = points are elements of a set
point-free = points are models of a geometric theory
pointwise = reason with points
pointless = reason without points
generalized = first-order geometric theories
ungeneralized = **localic** = propositional geometric theories

Geometric reasoning: pointwise treatment of point-free spaces

AUs: Algebraic approach to infinite disjunctions

Theory of bounded $\mathcal{S}$-toposes not cartesian ☹

$\mathcal{S}[\mathbb{T}]$ defined concretely (sheaves); characterized only up to equivalence.

AU-contexts sufficient in practice – eg $\mathbb{R}$ [MV12]

Use internal $\mathbb{N}$ to express countable joins in logic.

$[\mathbb{T}]$: notation for formal dual of $\mathbf{AU}\langle\mathbb{T}\rangle$ – "space of $\mathbb{T}$-models"

$\mathbb{T}$ gives site in any $\mathcal{S}$ with nno

$\mathcal{S}[\mathbb{T}]$: category of $\mathcal{S}$heaves with respect to $\mathcal{S}$ [Vic99]
2-functor $\mathcal{S}[-]\colon \mathfrak{Con} \to \mathfrak{BTop}/\mathcal{S}$

Choice of $\mathcal{S}$ now irrelevant! ☺

Get base-independent treatment of classifying toposes [Vic17], fibrationally over choice of base.

Dependent type = bundle = (base point $\mapsto$ fibre)

Context extensions

$$\begin{array}{ccc} \mathbb{T}_1 & & [\mathbb{T}_1] \\ \cup & & \downarrow p \\ \mathbb{T}_0 & & [\mathbb{T}_0] \end{array}$$

$\mathbb{T}_1$ is extension of $\mathbb{T}_0$
p is model reduction.

As bundle: base point x (in $\mathcal{S}$) $\mapsto$ fibre $\mathcal{S}[\mathbb{T}_1/x]$

$$\begin{array}{ccc} \mathcal{S}'[\mathbb{T}_1/f^*(x)] & \longrightarrow & \mathcal{S}[\mathbb{T}_1/x] \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{S}' & \xrightarrow{f} & \mathcal{S} \end{array}$$

x model of $\mathbb{T}_0$ in $\mathcal{S}$ (elementary topos with nno).
$\mathbb{T}_1/x$: $\mathcal{S}$-geometric theory of $\mathbb{T}_1$-models y with $p(y) = x$
$=$ substitution x for $\mathbb{T}_0$ in $\mathbb{T}_1$.

Construction preserved by bipullback along geometric morphisms [Vic17]

$=$ substitution $f^*(x)$ for x in $\mathbb{T}_1/x$ ... so DTT somewhere?

General examples: sites

Localic $\mathcal{S}$-toposes as fibres

$\mathbb{T}_0$ = theory of GRD-systems (frame presentations) [Vic04]

$\mathbb{T}_1$ = theory of GRD-systems equipped with point

Application: Powerlocale constructions on bundles work fibrewise. (Dependent types!) Represent powerlocale as endomap on $[\mathbb{T}_0]$.

General bounded $\mathcal{S}$-toposes as fibres

$\mathbb{T}_0$ = theory of sites

$\mathbb{T}_1$ = theory of sites equipped with continuous, flat presheaf

AU-logic has potential to address topos theory

? How to address continuous maps?
Not every space exponentiable, $\therefore$ can't classify them.
eg why are powerlocales functorial?

? Exploit internalization, use theory of AUs equipped with sites?

How to exploit the continuity of AU reasoning?

Some non-AU constructions on sets are intrinsically discontinuous!

Y^X Natural topology on Y^X is compact-open, not discrete.
If Y^X still locally compact, then set-theoretic Z^{Y^X} disagrees with topological answer.

$\Omega, \mathcal{P}X$ Similar. Topology is Scott.

$\mathcal{U}$ Universe $\mathcal{U}$ is just one discrete approximation (out of many) to the generalized space [set], the object classifier.
(cf. Garner [Gar12] *ionads* = point-set toposes.)
Topologically, any map out of [set] must be functorial and preserve filtered colimits. $[\mathrm{set}]^{[\mathrm{set}]}$ is the space of diagrams over Fin. $\mathcal{U}^{\mathcal{U}}$ won't be.

Two foundational approaches, from sets to spaces

Sets including arrow types

Start by allowing discontinuities.

Introduce *bureaucracy* to disallow them: eg point-set spaces, frames, sites, geometric reasoning.

Discontinuity can still appear in construction of sites – eg space of non-trivial rings apparently of presheaf type.
And often we don't know how else to do the topos theory!

AU-logic for sets

$\Rightarrow$ no bureaucracy needed for (point-free) spaces – no discontinuity to disallow.

Pointwise:
map is point $\mapsto$ point,
bundle is base point $\mapsto$ (theory for) space.

? dependent type theory of spaces.

Introduce discontinuity later – if you need it.

AU-mathematics?

Pros

- Pointwise reasoning for point-free spaces (cf. geometric techniqes)
- Fibrewise topology of bundles (base point $\mapsto$ fibre as point-free map)
- Hence dependent type theory of spaces

Cons

- Uses of eg arrow types must be justified
  eg Y^X as *space*
- Can we regain deep applications of topos theory? Known proofs often rely on discontinuous construction of sites.

The mathematics is different!

eg Excluded middle holds [MV12]

P a proposition, subset (= open subspace) of $1 = \{*\}$.

$\neg P$ $= \emptyset^P$ is *not a proposition* – it's a *closed subspace* of 1.
Its topology is Stone, not discrete – Boolean algebra of clopens

$$B = \mathrm{BA}\langle \quad \mid 0 = 1 \ (* \in P)\rangle.$$

Over P, B is degenerate, so no prime filters.

$\neg\neg P$ $= \emptyset^{\emptyset^P} = P$. BA hom from degenerate BA to B exists where P holds.

$\vee$ exists as *join of subspaces*.
$P \vee \neg P = \top$.

$\wedge$ exists too.
$P \wedge \neg P = \bot$

Conclusions

pt I:

- IMT for cartesian theories, straightforward term/congruence proof using partial logic [PV07]
- **AUs** as "sweet spot" for IMT [Mai05, MV12]

pt II:

- Algebraic approach to **classifying categories**, in particular classifying AUs [PV07]
- Now characterized up to isomorphism (not just equivalence).
- Strictness issues addressed by restricting AU-theories to "AU-contexts" [Vic19]

pt III:

- Usual constructive foundational approaches (elementary toposes, type theory) allow discontinuities, then use bureaucracy to disallow them.
- AU-logic intrinsically continuous (geometric).
- Vision: Do all continuous mathematics (of **generalized spaces**) in AU-logic without bureaucracy [Vic99, Vic17]
- Can that include deep applications of topos theory?

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