# Toposes pour les vraiment nuls

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June 1, 2009

**Abstract**

Restriction to geometric logic can enable one to define topological structures and continuous maps without explicit reference to topologies. This idea is illustrated with some examples and used to explain toposes as generalized topological spaces.

This is a reset and updated version (2009) of the 1996 article published as [Vic96b].

## 1 Introduction

Last year I wrote a short article [Vic96a], with the aim of explaining toposes (and in particular their nature as generalized topological spaces) to those who knew nothing about them. Unfortunately, it seems that I was unsuccessful in that aim, for when I presented the material in a seminar I was told that I should write another article *pour les vraiment nuls*. This is intended to be it.

The "generalized topological space" view is only one aspect of toposes, but it is an aspect that is very easily obscured. My aim here is to plant sufficiently strong intuitions that the reader can then turn to the standard texts and their technical development without losing sight of the generalized spaces.

I shall not give comprehensive references, but simply refer the reader to some standard introductory texts: for locales, [Joh82] and [Vic89]; for toposes, [MLM92], [McL91], [Bel88] and [Joh77] (though the last is hardly recommended for the beginner).[1] The topos books provide a good variety of approaches, but still tend to follow a broadly similar route governed by the view of toposes as generalized categories of sets. A number of my references to [MLM92] will therefore also apply to the other three.

## 2 Real numbers

For our leading example, let a *real number* be defined as a Dedekind section of the rationals, i.e. a pair $(L, R)$ of subsets of the rational numbers $\mathbb{Q}$, such that

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[1]Post-publication update: [Joh77] has now been superseded by [Joh02a] and [Joh02b]. See also [Vic07], which in effect provides a readers' guide to [MLM92] from the geometric point of view.

- $L$ is inhabited: $\exists q{\in}\mathbb{Q}.\ q{\in}L$
- $L$ is "rounded lower": $\forall q{\in}\mathbb{Q}.\ (q{\in}L{\leftrightarrow}\ \exists q'{\in}\mathbb{Q}.\ (q < q'{\wedge}q'{\in}L))$
- $R$ is inhabited and rounded upper
- $L$ and $R$ are disjoint
- $L$ and $R$ are "arbitrarily close": $\forall q,r{\in}\mathbb{Q}.\ (q < r{\rightarrow}q{\in}L{\vee}r{\in}R)$

(For a real number $x$, I shall usually write $(L_x, R_x)$ for the corresponding section.)

In a topological development, one would follow this with a definition of the usual *topology* on the set $\mathbb{R}$ of real numbers. If $q$ is rational, let us write $(q,\infty)$ for the set of reals $x$ for which $q{\in}L_x$ (i.e. $q < x$), and $(-\infty, q)$ for the set of reals $x$ for which $q{\in}R_x$. These are defined to be "subbasic open" subsets of $\mathbb{R}$, and in general the open subsets are those that can be expressed as unions of finite intersections of subbasics. (The intersection $(-\infty, q+\varepsilon){\cap}(q-\varepsilon,\infty)$ gives the rational open ball $B_\varepsilon(q)$, the set of reals strictly between $q-\varepsilon$ and $q+\varepsilon$, and the open sets are usually characterized as the unions of these.)

However, there is a good sense in which this topology is already inherent in the definition of the real numbers themselves. For the two sets $L$ and $R$ can be described equally well by two $\mathbb{Q}$-indexed families of propositions "$q{\in}L$" and "$q{\in}R$", and so we can equivalently present the theory of Dedekind sections as a propositional theory, with propositional symbol schemas $(q,\infty)$ and $(-\infty,q)$. The axioms now become the schemas

- $\mathbf{true}{\rightarrow}\bigvee_q(q,\infty)$ (note the use of an infinitary disjunction here)
- $(q',\infty){\rightarrow}(q,\infty)$ $(q < q')$
- $(q,\infty){\rightarrow}\bigvee_{q<q'}(q',\infty)$
- $\mathbf{true}{\rightarrow}\bigvee_q(-\infty,q)$
- $(-\infty,q'){\rightarrow}(-\infty,q)$ $(q > q')$
- $(-\infty,q){\rightarrow}\bigvee_{q>q'}(-\infty,q')$
- $(q,\infty){\wedge}(-\infty,q){\rightarrow}\mathbf{false}$
- $\mathbf{true}{\rightarrow}(q,\infty){\vee}(-\infty,r)$ $(q < r)$

It should be clear that these two theories are equivalent. Though there are technical differences concerning the precise form of a model (one theory uses subsets of $\mathbb{Q}$, the other uses $\mathbb{Q}$-indexed families of truth values), there should be no doubt that these differences are inessential.

We see now that the subbasic opens correspond exactly to the primitive propositional symbols, and $\cap$, $\bigcup$ and $\subseteq$ correspond to the logical $\wedge$ (finitary conjunction), $\bigvee$ (arbitrary disjunction) and $\rightarrow$. If we restrict our connectives to $\wedge$ and $\bigvee$ then the propositions we get correspond exactly to the opens. Of course, this appears to depend critically on the presentation of the theory. We could equally well present an equivalent theory using the analogues of the closed subsets $[q, \infty)$ and $(-\infty, q]$ (i.e. $\neg(q{\in}R)$ and $\neg(q{\in}L)$) and get a different topology. But plainly the problem there is that negation $\neg$ has crept into the presentation. If we restrict our connectives also in getting equivalences between presentations, then there is no problem – the topology is presentation invariant.

Let us continue by looking at continuous maps, taking as our example the negation map $\mathrm{neg} : \mathbb{R} \to \mathbb{R}$, defined by

$$L_{\mathrm{neg}(x)} = \{-q : q{\in}R_x\}, R_{\mathrm{neg}(x)} = \{-q : q{\in}L_x\}$$

It is easy to show that these give a new Dedekind section. However, this should normally be augmented by proof of continuity, to show that the inverse images of opens (or, sufficiently, subbasic opens) are open:

$$\mathrm{neg}^{-1}(q, \infty) = (-\infty, -q)$$

because

$$x{\in}\,\mathrm{neg}^{-1}(q, \infty) \Leftrightarrow \mathrm{neg}(x){\in}(q, \infty) \Leftrightarrow q{\in}L_{\mathrm{neg}(x)} \Leftrightarrow -q{\in}R_x \Leftrightarrow x{\in}(-\infty, -q)$$

and

$$\mathrm{neg}^{-1}(-\infty, q) = (-q, \infty)$$

But let us study more closely how the equivalence of the two theories works here. A Dedekind section $x$, i.e. a model of the predicate theory, yields interpretations of all the symbols $(q, \infty)$ and $(-\infty, q)$ as truth values. Negation constructs a new Dedekind section $\mathrm{neg}(x)$, and, correspondingly, new interpretations $(q, \infty)'$ and $(-\infty, q)'$ (say) of the propositional symbols: e.g. $(q, \infty)'$ is true iff $q{\in}L_{\mathrm{neg}(x)}$, i.e. $\mathrm{neg}(x)$ is in the open set corresponding to $(q, \infty)$. It follows that $(q, \infty)'$, considered as a subset of $\mathbb{R}$ (comprising those models x for which $(q, \infty)'$ is true) is exactly $\mathrm{neg}^{-1}(q, \infty)$, and similarly for $(-\infty, q)'$.

Now in constructing the new Dedekind section $\mathrm{neg}(x)$ from the original one, $x$, we find that, correspondingly, $(q, \infty)'$ and $(-\infty, q)'$ were constructed from the propositional symbols $(q, \infty)$ and $(-\infty, q)$. Hence continuity is obvious if the propositional constructions are constrained to using $\wedge$ and $\bigvee$. It turns out that this will follow, provided that the construction as described on the Dedekind sections also observes certain "geometric" constraints. We shall examine these in the next section.

# 3 Geometric logic and locales

The example of Section 2 is suggesting that if we restrict our logic suitably, then topological issues are completely implicit: the topology is implicit in the theory presentation that defines the points (as models), and continuity is implicit in the construction of the maps. The appropriate logic is *geometric logic*.

*To describe the points geometrically is already to topologize them.*

A propositional geometric theory is one presented using axioms of the form $\phi \rightarrow \psi$ where the propositions $\phi$ and $\psi$ are "geometric" – their connectives are restricted to $\wedge$ (finitary conjunction) and $\bigvee$ (arbitrary disjunction). The approach then is to describe the points of a topological space as the models of a propositional geometric theory, and the open sets as the geometric propositions (modulo equivalence). This works even if we replace the propositional theory by an equivalent non-propositional one.

Let us take the word *locale* to have the intuitive meaning of "space of models of a propositional geometric theory", in other words our revised idea of topological space. This is a little vague – what, for instance, is a "space"? But the idea is that if we can present a propositional theory $T$, then something or other should be the corresponding locale $[T]$, "the space of $T$-models".

But of course, the word "locale" already has a use, and the link with the intuitions above is via the notion of the *frame* $\Omega[T]$, the set of *all* expressible geometric propositions in $T$, modulo equivalence. The frame is effectively a geometric Lindenbaum algebra, and algebraically it is presented by generators and relations taken from $T$ [Vic89]. Moreover, the frame provides a canonical way of presenting the locale, so locales and their properties can be studied through frames and their properties: and of course this has been done extensively [Joh82].

The next point to make is that these ideas extend to continuous maps. If $X$ and $Y$ are two locales, then to define a map from $X$ to $Y$, continuous for the implicit topologies, it suffices to give a uniform description of the form, "Let $x$ be a point of $X$. Then $f(x)$ is defined [geometrically] to be . . . ."

*Continuity = uniformity + geometricity*

If we cheerfully accept the restrictions of geometric logic, then topology appears much simpler, much closer to sets and functions:

- To define the topological space, just give a geometric description of its points (no separate topology needed).
- To define a continuous map, just give a uniform, geometric description of how the result $f(x)$ is constructed from $x$ (no continuity proof needed).

An example of these two ideas in application is [Vic05].

If we refuse to relinquish the power of classical logic, then we enable ourselves to define equivalences between theories without respecting the topologies – so the topology is no longer presentation invariant. We also enable ourselves to define discontinuous functions (examples later) so that a separate continuity proof is needed: this is really an explicit warranty that we haven't misused the extra power. To put it another way, classical point-set topology is a complicated (and only partially successful) machinery whose purpose is to correct for the errors introduced by classical reasoning principles! To use an image from planetary orbits, geometric logic is ellipses, classical logic is circles and point-set topology is epicycles.

In Section 7 we shall consider more critically whether we really can just adopt a geometric logic and find topology doing itself automatically.

### 3.1 Specialization order

Though it's not evident in $\mathbb{R}$, locales automatically bear an ordering. If $x$ and $y$ are two points of a given locale, then it may be that every open (proposition) that is true for $x$ is also true for $y$. To put it another way, every open containing $x$ also contains $y$. We then say that $y$ *specializes* $x$, and write $x \sqsubseteq y$. On $\mathbb{R}$, this ordering is discrete ($x \sqsubseteq y$ iff $x = y$), but for most locales – including the domains used in denotational semantics – it is non-discrete and important. It is also true that for any locale we have directed joins (sups) of points. In denotational semantics, the importance of this is vital.

Any continuous map between locales is monotone with respect to the specialization order, essentially because of the positivity of geometric logic – the map cannot use any negative information about its argument. More subtly, because of the finiteness of conjunctions in geometric logic, the map must preserve directed joins.

This ordering appears quite concretely as an order enrichment on the category of locales. If $f$ and $g$ are two continuous maps from $X$ to $Y$, then $f \sqsubseteq g$ iff we have a proof of the form, "Let $x$ be a point of $X$. Then $f(x) \sqsubseteq g(x)$ (geometrically)."

## 4 More examples

We follow with three more examples. The first illustrates how classical reasoning can lead to discontinuous functions, the second gives a deeper flavour of how the geometric reasoning works in practice, and the third marks a distinction between geometric logic and intuitionistic logic.

### 4.1 The Heaviside step function

We consider the Heaviside function $H : \mathbb{R} \to \mathbb{R}$, defined by letting

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

In terms of Dedekind sections, we have $x < 0$ iff $0 \in R_x$, so

$$q \in R_{H(x)} \text{ (i.e. } q > H(x)) \text{ iff } q > 1 \vee (q > 0 \wedge 0 \in R_x)$$
$$q \in L_{H(x)} \text{ (i.e. } q < H(x)) \text{ iff } q < 0 \vee (q < 1 \wedge 0 \notin R_x)$$

To prove that $(L_{H(x)}, R_{H(x)})$ is a Dedekind section is largely straightforward, except that in the final axiom we must use excluded middle on the formula

$0\in R_x$. (We also use it on formulae concerning the order of rationals, but that is no problem for that order is decidable.) Suppose $q < r$. If $q < 0$ then $q\in L_{H(x)}$, and if $r > 1$ then $r\in R_{H(x)}$, so the only question is over the case $0 \leq q < r \leq 1$. But then if $0\in R_x$ we have $r\in R_{H(x)}$, while if $0 \notin R_x$ we have $q\in L_{H(x)}$.

## 4.2 Addition of reals

This example shows more clearly the influence of the geometric constraints on the reasoning: the influence is often quite algorithmic in nature. We can define addition on $\mathbb{R}$ as follows.

Let $x = (L_x, R_x)$ and $y = (L_y, R_y)$ be two Dedekind sections. Then their sum $x + y$ is defined by

$$L_{x+y} = \{q + r : q\in L_x, r\in L_y\}$$
$$R_{x+y} = \{q + r : q\in R_x, r\in R_y\}$$

It is required to show that $x + y$ is again a Dedekind section. Most of the axioms are easily disposed of, but the final one needs harder work.

**Lemma 1** *Let $(L, R)$ be a Dedekind section, and let $\varepsilon$ be a positive rational. Then there are rationals $q\in L$ and $r\in R$ such that $r - q < \varepsilon$.*

**Proof.** $L$ and $R$ are both inhabited, so we can find $q_0\in L$ and $r_0\in R$. Certainly $q_0 < r_0$. There is some natural number $n$ such that $2^n\varepsilon > r_0 - q_0$; the proof is by induction on $n$. If $n = 0$, then we are already done.

For the induction step, suppose $2^{n+1}\varepsilon > r_0 - q_0$ and let the interval $[q_0, r_0]$ be divided into four equal parts by $q_0 < s < t < u < r_0$. Out of these, we find numbers $q_1\in L$ and $r_1\in R$ such that $r_1 - q_1 = (r_0 - q_0)/2$ and we can then use the induction hypothesis. We have

$$s\in L \vee t\in R \text{ and } t\in L \vee u\in R$$

If $t\in R$, we define $q_1 = q_0$, $r_1 = t$. If $t\in L$, then $q_1 = t$, $r_1 = r_0$. The remaining case has $s\in L$ and $u\in R$, and then we can take $q_1 = s$, $r_1 = u$. In all three cases we have $q_1$ and $r_1$ as required.

Note the algorithmic flavour of this: we start off with rationals $q_0$ and $r_0$ in $L$ and $R$, and then an iteration replaces these by two closer rationals in $L$ and $R$. We iterate this until they are close enough, when we can halt with the answer. To be sure, it is highly non-deterministic – the algorithm will usually offer a choice of new numbers. But the logic ensures that the choice is non-empty. ■

Returning to the problem of sums, suppose $s < t$. By the Lemma, we can find $q_x\in L_x$ and $r_x\in R_x$ such that $r_x - q_x < t - s$. Then $s - q_x < t - r_x$, so we have either $s - q_x\in L_y$, in which case $s\in L_{x+y}$, or $t - r_x\in R_y$, in which case $t\in R_{x+y}$.

## 4.3 Upper real numbers

Brouwer maintained that all functions on the real line were continuous: that "constructions" of discontinuous functions must inevitably use some inadmissible principles, for instance of "omniscience". Our previous section can clearly be understood as suggesting that the admissible principles and constructions, that always lead to continuous functions, are the geometric ones. That general argument applies to far more than just the real line. However, Brouwer's ideas led to Heyting's intuitionistic logic, and what I want to present in this section is an example, not quite of the real line, where even intuitionistic reasoning can lead to discontinuous functions.

Let us define an *upper real* to be an inhabited rounded upper subset of $\mathbb{Q}$ – we shall normally write $R_x$ for the subset itself, $x$ for the upper real considered more abstractly. Classically, $R_x$ is either the whole of $\mathbb{Q}$ or the second component of a Dedekind section, so the space of upper reals could be denoted $[-\infty, \infty)$. However, the natural topology has a basis comprising the sets $(q, \infty)$, and its specialization order is numerical $\geq$ (so big numbers are low in the specialization order). To suggest that, let us write it as $\overleftarrow{[-\infty, \infty)}$. (Note the fact that $-\infty$ has appeared. For any locale we have all directed joins of points, so $-\infty$ has to be there as the directed join of the points in $(-\infty, \infty)$ under the specialization order $\geq$.) The example we shall give actually concerns the sublocale $\overleftarrow{[0, \infty)}$, whose points are inhabited rounded subsets of the set of *positive* rationals.

Let $x$ and $y$ be two points of $\overleftarrow{[0, \infty)}$. Then their truncated difference, $x \dot{-} y$, is defined by

$$q \in R_{x \dot{-} y} \text{ iff } \exists q' \in \mathbb{Q}.\ 0 < q' < q \wedge \forall r \in R_y.\ q' + r \in R_x$$

It is not hard to show that this is rounded upper and contains only positive rationals. As for inhabitedness, let $q' \in R_x$. Then any $r$ in $R_y$ is positive, so $q' + r$ is also in $R_x$. It follows that $q' + 1$ is in $R_{x \dot{-} y}$.

This is truncated difference, i.e. $x \dot{-} y = \max(0, x - y)$, and in fact it is possible to justify this by proving –

**Proposition 2** *For any $x, y, z$ in $\overleftarrow{[0, \infty)}$ we have $z \geq x \dot{-} y$ iff $z + y \geq x$.]*

Now the interesting part of this is that all the reasoning – in both the construction and the Proposition – is intuitionistically valid, but truncated minus is not continuous. This is obvious, because it is antitone in its second argument – if $y \leq y'$ then

$$(x \dot{-} y) + y' \geq (x \dot{-} y) + y \geq x,$$

so $x \dot{-} y \geq x \dot{-} y'$ – and continuous maps must be monotone (with respect to the specialization order). We can see that the construction is not geometric, because of the universal quantification $\forall r \in R_y$.

# 5 Predicate theories and toposes

The locales correspond to propositional geometric theories, but there are also more general predicate theories, not all equivalent to propositional ones. The "space of models" for one of these will not in general be a locale. Nonetheless, our programme of accepting the geometric discipline still makes perfect sense, and this is where we shall use the word *topos*: a topos is the "class of models" (its points) for a general geometric theory, and a continuous map is a geometrically constructed transformation of points. (The topos is usually referred to as the *classifying topos* for its theory, and such a continuous map is usually known as a *geometric morphism*.) Very literally, a topos is a generalized locale (a locale is simply a topos whose corresponding geometric theory happens to be propositional), and so, as Grothendieck said, "A topos is a generalized topological space." In accepting the geometric discipline we are therefore again, in some generalized sense, doing topology without being aware of it.

## 5.1 "Geometric mathematics"

I should first say something about the nature of these geometric constructions.

*Logically*, we get a restriction on the theories presented: a geometric theory is a many-sorted first order theory, with a vocabulary of sorts, predicates and functions, and axioms of the form $\phi \vdash_S \psi$ where $\phi$ and $\psi$ are geometric formulae (their permitted connectives are $\wedge$, $\bigvee$, $=$ and $\exists$) whose free variables are all taken from the finite set $S$.

*Set theoretically*, we get a restriction on the constructions allowed. The general rule is that the geometric constructions are those that can be characterized uniquely up to isomorphism by geometric theories. This includes finite products, arbitrary coproducts (disjoint unions), equalizers and coequalizers, and also free algebra constructions (for finitary algebraic theories). This gives us list types (free monoids), finite powersets (free semilattices) as well as $\mathbb{N}$ (natural numbers) and $\mathbb{Z}$ (integers), from which $\mathbb{Q}$ (rationals) can be constructed. However, certain constructions are *not* geometric, notably power sets, function sets and $\mathbb{R}$. (Geometrically, we can define a theory whose models are the reals, but we cannot define the set of reals: so geometrically, $\mathbb{R}$ is a locale but not a set.)

The practical mathematical effect is to allow finitary constructions, as well as inductive and recursive ones, but not impredicative ones. This gives geometric mathematics a highly algorithmic flavour, building constructions – and often proofs too – "from below".

In presenting theories we might as well presume a geometric type theory, from which we can derive new types from the base types (the sorts). We have already seen this in the theory of Dedekind sections. There were no base types there, but out of nothing we can construct $\mathbb{Q}$, which we used there, and hence $\mathbb{Q} \times \mathbb{Q}$ and so forth. Furthermore, by using the finite power set construction we get a weak 2nd order logic in which finite subsets can appear as terms, and we can universally quantify over finite sets despite the fact that $\forall$ is not in general geometric.

We now have a first view of toposes, based on the analogy with locales:

- A topos is defined by giving a geometric theory of its points.
- To define a geometric morphism $f$ from $E$ to $F$, just give a uniform, geometric description of how the result $f(x)$ is constructed from $x$.

Already this is sufficient to do a certain amount of topos theory, and this can be seen quite explicitly in [Vic99] which reworks certain categories and functors of domain theory as toposes and geometric morphisms.

### 5.2 Homomorphisms

The specialization order on locales generalizes to a category structure on a topos, through the notion of *homomorphism* between two models: it comprises, for each base type, a function between the two corresponding carriers, that respects the functions and relations. (e.g. Suppose the theory has one base type, a unary function $f$, and a binary relation $R$. If $\phi$ is the carrier function for a homomorphism, then we require $f(\phi(x)) = \phi(f(x))$ and $R(x, y) \Rightarrow R(\phi(x), \phi(y))$.) Because of the positivity of the logic, such a homomorphism can be defined at all the derived types too, and the notion of homomorphism is presentation-independent. In the propositional case (no sorts), such a homomorphism is precisely an instance of the specialization order.

The good behaviour of homomorphisms means that every topos has an intrinsic category structure: object = point, morphism = homomorphism. Moreover, any filtered diagram of points (such as a chain $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$) has a colimit. A map (geometric morphism) between toposes is functorial with respect to homomorphisms and preserves the filtered colimits.

The homomorphisms make the category of toposes into a 2-category. For if $f$ and $g$ are two continuous maps (geometric morphisms) between toposes $X$ and $Y$, then we can define 2-cells (or natural transformations) $\alpha : f \Rightarrow g$ in the form, “Let $x$ be a point of $X$. Then $\alpha_x$ is a homomorphism from $f(x)$ to $g(x)$ defined [geometrically] as . . . .”

## 6 Generalized categories of sets

The connection between propositional geometric theories and topologies was not too strained, and was made all the firmer by the frames. But for the predicate theories there is no pre-existing “generalized topology” we can connect up with, and the best we can do is to find something corresponding to the frames. For a locale $D = [T]$, its frame $\Omega D$ comprises all the propositions expressible in $T$. Now propositional truth values are the points of a locale \$ (the *Sierpinski* locale, presented by a single propositional symbol and no axioms), and it follows that the elements of $\Omega D$ are the maps from $D$ to \$.

For a *topos* $D$, classifying a predicate theory, the propositions are – unsurprisingly – not enough. We must replace \$ by [set], the topos classifying sets, and define $\mathcal{S}D$ to be the class of geometric morphisms from $D$ to [set].

**Example 3** *1. Suppose $D$ is a locale. $\mathcal{S}D$ is the* category of sheaves *over $D$, and definitions (which can now be understood as structural elucidations) can be found in the standard texts such as [MLM92]. Other intuitions are in [Vic96a]. By extension, we shall refer to objects of $\mathcal{S}D$ as* sheaves on *$D$ even when $D$ is not a locale. If $F$ is a sheaf on $D$, and $x$ is a point of $D$, then $F(x)$ is the* stalk of $F$ at $x$.

2. *Suppose $D$ is the locale classifying the empty theory – no vocabulary, no axioms. This theory has a unique model given by the vacuous interpretation of the empty vocabulary, so the locale has a unique point and is written* 1. *$\mathcal{S}1$ is the category of sets.*

3. *Suppose $D$ is [set], and write $FC$ for the category of finite cardinals. These are points of [set], so we get a functor from $FC \times \mathcal{S}[set]$ to* **Sets***, and hence one from $\mathcal{S}[set]$ to the functor category $\mathbf{Sets}^{FC}$. It turns out (e.g. [MLM92]) that this is in fact an equivalence: a sheaf on [set] is determined by its stalks at finite cardinals, the essential reason being that an arbitrary set is a filtered colimit of finite cardinals. This behaviour (that $\mathcal{S}D$ is equivalent to $\mathbf{Sets}^C$ where $C$ is some restricted category of points) is typical of toposes that classify algebraic theories.*

The objects of $\mathcal{S}D$, the sheaves, are essentially parametrized sets, and for this reason $\mathcal{S}D$ behaves in many respects just like a (non-classical) category of sets. It has all the geometric set-theoretic constructions (finite products, arbitrary coproducts, etc.), and indeed it has some non-geometric ones too (principally, function sets and power sets): it is a *generalized category of sets*. It certainly has enough good properties that we can say what it means to find a model in $\mathcal{S}D$ for a geometric theory.

We have now found a wide range of generalized categories of sets (indeed this is the key to the logical applications of topos theory), and in each of these we can discuss models of a theory $T$. It is important to realise that when we refer to points of the topos $[T]$, we allow ourselves to consider models of $T$ in arbitrary generalized categories of sets, not just some favourite "underlying set theory".

Purely in terms of our definition of $\mathcal{S}D$, it is actually not too hard to see –

**Theorem 4** *Let $D$ and $E$ be toposes. Then the following are equivalent:*

1. *Geometric morphisms from $D$ to $E$.*
2. *Points of $E$ in $\mathcal{S}D$.*
3. *Functors from $\mathcal{S}E$ to $\mathcal{S}D$ that preserve geometric constructions.*

**Proof.** *(Hint:)* Use the fact that $\mathcal{S}D$ contains a "generic" point of $D$. For instance, given a sort in a theory $T$, what is the corresponding carrier for the generic model? It is an object of $\mathcal{S}[T]$, a sheaf on $[T]$: its stalk at point $x$ (model of $T$) is the corresponding carrier of $x$. ∎

A consequence of this is that, just as with locales and frames, toposes and their properties can be studied through the categories of sheaves and their properties, which are moreover interesting in their own right. It has therefore become customary to say that the topos is its category of sheaves (and use the phrase "elementary topos" for some even more generalized categories of sets), but this does not harmonize with the idea of generalized spaces. For instance, the category structure of $\mathcal{S}D$ – sheaves and sheaf morphisms – is completely different from the category structure of $D$ – points and homomorphisms.

# 7 How much does Geometric Mathematics encompass?

The propaganda you've seen above – Do all your mathematics geometrically and you need not think about topology or frames or categories of sheaves – is in many ways delightfully simple. It holds out the promise of a grand encompassing geometric mathematics in which topology and continuity are intrinsic features, not added structure. Let us consider the claim a little more soberly.

It works quite well when defining the spaces and the maps, but it is heuristically naive for anything deeper. As an example, let us consider compactness. There is a good geometric account of this, but it is non-obvious and requires some structural knowledge of geometric mathematics itself. How far similar methods can be applied to other topological concepts such as connectedness and completeness is not entirely clear.

Compactness is known from ordinary topology and can readily be transferred to frames; the question is whether it can be discussed purely in terms of locales, given by theory presentations. The trick is to use the upper powerlocale $P_U D$ ([Vic95], [Vic97]). The points of $P_U D$ are certain compact sublocales of $D$, with reverse ordering (big sublocales are low in the specialization order), and it turns out that $D$ itself is compact iff $P_U D$ is local (has a bottom point). Now the presentation for $P_U D$ is normally derived from the frame $\Omega D$, but it turns out that it can always be derived, and by geometric constructions, direct from a presentation of $D$. Hence compactness comes out of a deep structural feature of geometric mathematics, namely the existence of the upper powerlocale construction.

Even in the localic context we have a non-trivial piece of work here. I at least am still far from understanding the analogous structure for toposes.

## 8 Acknowledgements

I should like to thank Sofia Jourdan for her careful review of the first draft of this paper, and the EPSRC for their support via the "Foundational Structures in Computer Science" project.

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