Continuous maps

X, Y topological spaces

$f: X \to Y$ a function

f is continuous if –

for every $V \subseteq Y$ open, $f^{-1}(V)$ is open

$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ inverse image

For any function f,

$f^{-1}: \mathcal{P}Y \to \mathcal{P}X$ preserves all unions & intersections

Continuity says it restricts to

$f^{-1}: \Omega Y \to \Omega X$

ΩX and ΩY are both frames

f^{-1} is a frame homomorphism

For each continuous map $f: X \to Y$ there is a frame homomorphism

$f^{-1}: \Omega Y \to \Omega X$

Logical view of continuous maps

(G_1, R_1), (G_2, R_2) two geometric theories

Models of theory are like points of a space.

How might you transform models of (G_1, R_1) to models of (G_2, R_2)?

Given: model M of (G_1, R_1)

truth value $M(g)$ for each $g \in G_1$ respecting axioms R_1

How to construct model N of (G_2, R_2) out of it?

Need truth values $N(h)$ $(h \in G_2)$

IDEA: **Do everything geometrically**

– to preserve affirmativity

$M(g)$'s all supposed to be affirmative

Define each $N(h)$ as disjunction of finite conjunctions of $M(g)$'s

Then $N(h)$'s affirmative too.

And do it uniformly

– same construction works for all models.

In terms of Lindenbaum algebras

Disjunctions of finite conjunctions of g's are (modulo $\Leftrightarrow$) elements of $Fr\langle G_1|R_1\rangle$

$\therefore$ Want function $G_2 \xrightarrow{\alpha_0} Fr\langle G_1|R_1\rangle$

Says for each $h \in G_2$, how its truth value $N(h)$ in N is constructed from the $M(g)$'s.

But want N to be a model

$\therefore$ α_0 must respect relations R_2

α_0 is a generalized model of (G_2, R_2) in $Fr\langle G_1|R_1\rangle$

- and we know these correspond to frame homomorphisms

$$Fr\langle G_2|R_2\rangle \longrightarrow Fr\langle G_1|R_1\rangle$$

Can sensibly <u>define</u> -

Geometric transformations of models of (G_1, R_1) to models of (G_2, R_2)

and frame homomorphisms

$$Fr\langle G_2|R_2\rangle \xrightarrow[\alpha]{} Fr\langle G_1|R_1\rangle$$

Action on models.

Ordinary (ungeneralized) model of (G_1, R_1)

- is frame homomorphism $Fr\langle G_1|R_1\rangle \xrightarrow{M} \Omega$
$\Omega = P1$

Compose with α -

$$Fr\langle G_2|R_2\rangle \xrightarrow{\alpha} Fr\langle G_1|R_1\rangle \xrightarrow{M} \Omega$$

Get model of (G_2, R_2)

Works just the same with generalized models

$$Fr\langle G_2|R_2\rangle \xrightarrow{\alpha} Fr\langle G_1|R_1\rangle \xrightarrow{M} A$$

Important case: <u>generic</u> model

$$Fr\langle G_2|R_2\rangle \xrightarrow{\alpha} Fr\langle G_1|R_1\rangle \xrightarrow{Id} Fr\langle G_1|R_1\rangle$$

- Take generic model in $Fr\langle G_1|R_1\rangle$
- Apply model transformer
- Get model of (G_2, R_2) in $Fr\langle G_1|R_1\rangle$
- That's the frame homomorphism.

Non-standard truth values

IDEA: To define f transforming models of (G_1, R_1) to models of (G_2, R_2)

Say: Let M be a model of (G_1, R_1)

Define $f(M)$, a model of (G_2, R_2)

but do it geometrically

Let M be a model of (G_1, R_1)
- Now have new truth values $M(g)$
- and derived truth values $(\vee, \wedge)$
- We are in a mathematical world where truth values are $Fr\langle G_1 | R_1 \rangle$ instead of Ω
- M is the generic model
- We can still do mathematics in this world if we're careful (work geometrically)
- Define $f(M)$

Back in the standard world what was a single model $f(M)$ inside the box

becomes a frame homomorphism outside

continuity automatic!

Locales

"Frames pretending to be topological spaces"

A locale X is equipped with a frame ΩX

A continuous map $f: X \to Y$ between locales

(or just map)

is a frame homomorphism $\Omega f: \Omega Y \to \Omega X$

X is ~~presented~~ by a geometric theory (G, R) if ΩX is $Fr\langle G | R \rangle$

Two categories

Fr - frames, frame homomorphisms

Loc - locales, continuous maps

$Loc = Fr^{op}$

Language for locales
- use language appropriate to spaces
- avoid any hint that a locale "is" its frame
- instead, a locale X "has" a frame ΩX

Trivial locales - 1

Locale 1 is presented by $(\emptyset, \emptyset)$
- no generators
- no axioms

Just one model!, available for free

$$\Omega 1 = Fr\langle \emptyset \mid \emptyset \rangle = \Omega$$

Proof: Ω is initial frame
- for any frame A, $\exists$ unique homomorphism $\Omega ! : \Omega \longrightarrow A$.

Uniqueness: If $p \in \Omega$ (p a truthvalue)
then $p = \bigvee \{ true \mid p \}$

$\therefore \Omega ! (p) = \bigvee \{ 1_A \mid p \}$

Existence: Must show ! a homomorphism.

e.g. $\Omega ! (p) \wedge \Omega ! (q) = \bigvee \{ 1_A \mid p \} \wedge \bigvee \{ 1_A \mid q \}$

$= \bigvee \{ 1_A \mid p \wedge q \}$ by distributivity

$= \Omega ! (p \wedge q)$

<u>Note</u> Proof may look unexpected.
But have avoided saying every truth value is either true or false.

Corollary 1 is final locale.
Unique map $! : X \longrightarrow 1$.

Trivial locales - 0

The locale 0 is presented by theory
$(\emptyset, \{ true \Rightarrow false \})$

<u>No</u> models.
- exactly one interpretation of generators (nothing needs to be interpreted)
- axiom requires that true, which can be affirmed in any model, is impossible.

$\therefore$ models are impossible.

$\Omega 0$ has exactly one element
Axiom implies top $\leq$ bottom
$\therefore$ all elements equal.

0 is initial locale
- because the singleton frame $\Omega 0$ is obviously final.

Points of locales

Suppose X presented by theory (G,R)

Model of theory

$= $ homomorphism $Fr\langle G|R\rangle \longrightarrow \Omega$
($Fr\langle G|R\rangle = \Omega X$, $\Omega = \Omega 1$)

$=$ map $1 \longrightarrow X$

For any locale X, define point of X is map $1 \to X$.

Generalized points

X, Y two locales, Y presented by (G,R)

Generalized model of (G,R) in ΩX

$=$ homomorphism $Fr\langle G|R\rangle \longrightarrow \Omega X$
($Fr\langle G|R\rangle = \Omega Y$)

$=$ map $X \longrightarrow Y$

For any Y, define generalized point of Y (at stage X) is a map $X \longrightarrow Y$

Think of it as a point of Y "parametrized by" points of X.

Non-classical reasoning

Generalized points are useful.

e.g. defining $f: X \longrightarrow Y$

> Let x be a point of X
>
> ⋮
>
> $f(x)$ defined as ...

Inside the box –

- working with generalized points at stage X
- x is generic point of X, $\mathrm{Id}: X \to X$
- $f(x)$ is a generalized point of Y.
- have a mathematics in which ΩX provides non-standard truth values

There's a price to pay

- not all classical mathematics is valid
- e.g. don't have "excluded middle" (every proposition either true or false)
- need intuitionistic reasoning to make it work at all
- weaker geometric reasoning to transfer properly between stages
- e.g. so construction on generic point works for all generalized points
- Must understand predicate geometric logic.