Example Real numbers $\mathbb{R}$

NOT geometric — NOT an observational set

BUT $\mathbb{R}$ is a locale

- propositional geometric theory whose models are the real numbers
- but easier to use equivalent predicate theory, essentially propositional = equivalent to a propositional theory

Propositional — no sorts
- so no variables, terms, $=, \exists$
- no arguments for predicates

Predicates = propositional symbols

Formulas — use $\wedge$ $\bigvee$

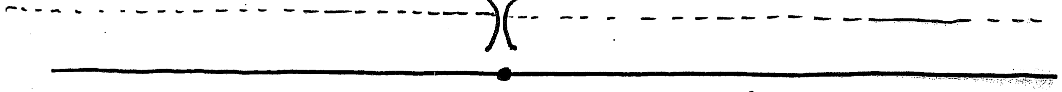
Essentially propositional — no declared sorts
- only sorts used are constructed out of nothing
- e.g. $\mathbb{N}$, $\mathcal{F}(\mathbb{N}\times\mathbb{N})$, ...

Dedekind sections

What is a real number?

One idea: section of rationals

L − rationals $< x$ R − rationals $> x$



x − a real

[Variants on idea use $\leq, \geq$ instead of $<, >$. But they don't give geometric theory.]

- $L = \{q \in \mathbb{Q} \mid q < x\}$ is −
  - lower closed − if $q < q' \in L$ then $q \in L$
  - inhabited − there is some $q \in L$
  - rounded − if $q \in L$ then $\exists q' > q$ with $q' \in L$
- $R = \{r \in \mathbb{Q} \mid r > x\}$ is inhabited and rounded ~~lower closed~~.
- L, R are disjoint
  Corollary: if $q \in L, r \in R$ then $q < r$
- If $q < r$ are rationals then either $q \in L$ or $r \in R$
  Corollary: if $\varepsilon \in \mathbb{Q}$, $\varepsilon > 0$ then $\exists q \in L, r \in R$ with $r - q < \varepsilon$
  (L, R come arbitrarily close together)

Geometric theory

name of theory

$$\mathbb{R}$$

$L(q) \subseteq \mathbb{Q}$ — sort constructed out of nothing

$R(r) \subseteq \mathbb{Q}$

$\forall q, q' \in \mathbb{Q}. (q < q' \wedge L(q') \to L(q))$

$true \to \exists q \in \mathbb{Q}. L(q)$

$\forall q \in \mathbb{Q}. (L(q) \to \exists q' \in \mathbb{Q}. (q < q' \wedge L(q')))$

$\forall r, r' \in \mathbb{Q}. (r' < r \wedge R(r') \to R(r))$

$true \to \exists r \in \mathbb{Q}. R(r)$

$\forall r \in \mathbb{Q}. (R(r) \to \exists r' \in \mathbb{Q}. (r' < r \wedge R(r')))$

$\forall q \in \mathbb{Q}. (L(q) \wedge R(q) \to false)$

$\forall q, r \in \mathbb{Q}. (q < r \to L(q) \vee R(r))$

Note: In practice don't use L, R.

In a given model x, write

$q < x$ for $L(q)$

$x < r$ for $R(r)$.

Transforming to propositional

Predicate $L(q)$ $(q \in \mathbb{Q})$ becomes $\mathbb{Q}$-indexed family of propositions

$L(q)$ true in a model $x \Leftrightarrow q < x < \infty$

∴ write (q, ∞) for proposition

$R(r)$ translates to propositions $(-\infty, r)$.

Predicate axiom:

$\forall q, q' \in \mathbb{Q}. (q < q' \wedge L(q') \to L(q))$

translates to family of propositional axioms

$(q', \infty) \to (q, \infty) \qquad (q < q')$

Can use $\bigvee$ for $\exists$:

$true \to \exists q \in \mathbb{Q}. L(q)$

translates to

$true \to \bigvee_{q \in \mathbb{Q}} (q, \infty)$

Equivalent propositional theory

$\mathbb{R}$

$(q,\infty) \le 1 \qquad (q \in \mathbb{Q})$

$(-\infty,r) \le 1 \qquad (r \in \mathbb{Q})$

$$(q',\infty) \to (q,\infty) \qquad (q<q')$$

$$\text{true} \to \bigvee_{q\in\mathbb{Q}} (q,\infty)$$

$$(q,\infty) \to \bigvee_{\substack{q'\in\mathbb{Q}\\ q<q'}} (q',\infty)$$

$$(-\infty,r') \to (-\infty,r) \qquad (r'<r)$$

$$\text{true} \to \bigvee_{r\in\mathbb{Q}} (-\infty,r)$$

$$(-\infty,r) \to \bigvee_{\substack{r'\in\mathbb{Q}\\ r'<r}} (-\infty,r')$$

$$(q,\infty)\wedge(-\infty,q) \to \text{false}$$

$$\text{true} \to (q,\infty)\vee(-\infty,r) \qquad (q<r)$$

Exercise -
1. Same models as previous presentation
2. Holds spatially when $\to, \wedge, \vee$ become $\subseteq, \cap, \cup$
3. Equivalent to p.123 'stone spaces.

Addition

Let x, y be real numbers.
Then $x+y$ is defined by -

$x+y<r$ iff $\exists\, r_x, r_y \in \mathbb{Q}$ s.t.

$$x<r_x \qquad y<r_y$$
$$r_x+r_y \le r$$

$q<x+y$ iff $\exists\, q_x, q_y \in \mathbb{Q}$ s.t.

$$q_x<x \qquad q_y<y$$
$$q \le q_x+q_y$$

"$x+y<r$" means $R(r)$ } in model $x+y$
"$q<x+y$" means $L(q)$ }

R - rounded upper inhabited

- Find r_x, r_y with $x<r_x$, $y<r_y$ - then $x+y<r_x+r_y$ } inhabited
- If $x+y<r\le r'$ then obviously $x+y<r'$ | upper
- If $x+y<r$ using r_x, r_y then $\exists\, r_x', r_y'$. $x<r_x'<r_x \quad y<r_y'<r_y$
  $\therefore x+y<r_x'+r_y'<r$ rounded

- L - rounded lower inhabited
  - similar.

- Suppose

$$x+y<s \quad \text{and} \quad s<x+y$$

$$x<r_x \quad y<r_y \qquad r_x+r_y \le s$$

$$\underline{q_x<x \quad q_y<y} \qquad s \le q_x+q_y$$

$$\Downarrow$$

$$q_x<r_x \quad q_y<r_y$$

$$\therefore\ s \le q_x+q_y < r_x+r_y \le s$$

- contradiction.

- Suppose $q<r$. Let $\varepsilon = \frac{r-q}{2}$

Find

$$q_x<x<r_x \qquad r_x-q_x<\varepsilon$$

$$q_y<y<r_y \qquad r_y-q_y<\varepsilon$$

Then $q_x+q_y<r_x+r_y \qquad r_x+r_y-q_x-q_y<\underset{=r-q}{2\varepsilon}$

$$\therefore\ q<q_x+q_y \quad \text{or} \quad r_x+r_y<r$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$q<x+y \qquad\qquad x+y<r$$

Propositionally

Addition

Frame homomorphism

$$\Omega + : \Omega\mathbb{R} \longrightarrow \Omega(\mathbb{R}\times\mathbb{R})$$

$$(-\infty,r) \longmapsto \bigvee_{r_x+r_y\le r} (-\infty,r_x)\times(-\infty,r_y)$$

$$(q,\infty) \longmapsto \bigvee_{q\le q_x+q_y} (q_x,\infty)\times(q_y,\infty)$$

Must check propositional axioms respected

☺ Don't have to question what is geometric

☹ Mathematics obscured by propositional formalism

LOCALES : SUMMARY

Locale = space
described and topologized
by propositional geometric theory
point = model
open = proposition

Equivalently:
can use essentially propositional geometric theory

- predicate logic
- no declared sorts
- but sort constructors still available

Continuous map
= model transformation
- constructed geometrically

Further topics

- Different kinds of spaces localically
  e.g. domains
- Topological properties
  e.g. compactness, openness ...
- Sublocales
- Different kinds of maps
  open, proper, perfect, ...
  surjection, inclusion
- Powerlocales (cf. powerdomains)
  - "coverage theorems"