Modelling Predator-Prey Interactions with ODE

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Module 06-23836: Computational Modelling with MATLAB

Outline of Topics

Predator-Prey Models

The Lotka-Volterra (LV) model

The Rosenzweig and MacArthur (RM) model

Phenomenon: Canadian lynx and snowshoe hares

All began with a hairy history: Hudson Bay Company fur trapping records

The number of furs purchased at the Company’s forts was meticulously recorded, for well over 100 years.

Phenomenon: Canadian lynx and snowshoe hares

The following figure shows the numbers of snowshoe hares, and one of their main predators, the Canadian lynx.

As we can see from the figure, there are clearly predator-prey cycles. Peaks and valleys at roughly 8-10 year intervals.
Identification: Questions

What factor(s) caused:

- The two populations oscillated cyclically?
- The cycle of predator population always lagged behind the prey population?

The Lotka-Volterra model

- Also known as the (simplest) predator-prey equations.
- Frequently used to describe the dynamics of biological systems in which two species interact, one a predator and one its prey.
- Also used in economic theory, e.g., model the interactions of two industrial sectors.
- The main hypothesis: The prey-predator interaction is the only factor causes:
  - The two populations oscillated cyclically.
  - The cycle of predator population always lagged behind the prey population.

Definition: General assumptions of the LV model

- The prey population is sustainable, e.g., they have ample food.
- The predator eats nothing else but the prey.
- The change rate of population is proportional to its size.
- The environment is static, which does not affect the change rate of population.

Definition: Specific assumptions of the LV model

- If no predators are present, the number of prey grows exponentially.
- Without prey to eat, the predator population declines at a rate proportional to its size.
- The rate at which the prey are eaten is proportional to the rate at which the predators and prey interact.
- The rate at which the predators are born is proportional to the rate at which the prey are eaten by the predator.

Definition: the equations

\[
\begin{align*}
\frac{dx}{dt} &= x(\alpha - \beta y) \\
\frac{dy}{dt} &= -y(\gamma - \delta x)
\end{align*}
\]

where \( y \) is the number/density of some predators, e.g., foxes; \( x \) is the number/density of its prey, e.g., rabbits; \( t \) represents the time and \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) represent the growth of the two population against time; \( \alpha, \beta, \gamma \) and \( \delta \) are parameters.

Can you relate those parameters to the assumptions?
**Definition:** The LV model in MATLAB

1. function dy = Lotka_Volterra_Model(t,y)
2. dy = zeros(2,1);
3. alpha = 0.9; beta = 1;
4. gamma = 0.2; delta = 0.3;
5. A = [alpha - beta*y(2), 0; 0, -gamma + delta*y(1)];
6. dy = A*y;
7. end

**Step 2:** Call this function using a numerical solver `ode45`

1. t0 = 0;
2. tfinal = 300;
3. % We normalise the initial condition
4. y0 = [10; 1];
5. % Simulate the differential equation.
6. [t,y] = ode45('Lotka_Volterra_Model',[t0 tfinal],y0);

**Simulation:** Let's run the model

- The results captured the essences of what we observed from the Hudson Bay Company fur records.
- The model is robust to a wild range of parameters and initial conditions.
- However, if we try a extreme initial condition: y0 = [1000; 1];
**Satisfied? No! Problems with the LV model**

- The cycle’s amplitude depends on the initial conditions, e.g., numbers of predators and prey, and the results are not realistic.
- Two problems:
  - It is assumed that the prey population grows exponentially indefinitely in the absence of predators.
  - The rate at which predators remove prey is not realistic.
- It is famous but not very successful, neither in the laboratory nor in the field...

**First problem with the LV model**

- In the absence of predators: \( y = 0 \). The first equation becomes:
  \[
  \frac{dx}{dt} = \alpha x
  \]
- That is the growth rate of prey \( x \) at time \( t \) is proportional to the value of \( x(t) \). Let’s assume the initial value \( x(0) = a \) and \( a > 0 \)
- Then we can solve the equation:
  \[
  \frac{dx}{x} = \alpha dt \Rightarrow \int \frac{dx}{x} = \int \alpha dt \Rightarrow \ln x = \alpha t + \text{constant}
  \]
- Incorporating the initial value gives:
  \[
  \ln x = \alpha t + \ln a \Rightarrow x = ae^{\alpha t}
  \]

**Second problem with the LV model**

- The rate at which predators remove prey is \( \beta xy \).
- The rate of an individual predator consume prey is \( \beta xy/y = \beta x \).
- A rate that continues to increase as \( x \) increases, no matter how large \( x \) becomes.
- In effect, this implies that an individual predator can never get full.

**The Rosenzweig and MacArthur (RM) model**

In order to address the problems of LV model, Rosenzweig and MacArthur proposed their model in 1963. This is a more realistic and general model of predator-prey interaction.

**Definition: Corrections in RM model**

To address the first problem, the RM model assumes:
- In absence of predator, the prey population grows according to the logistic growth equation:
  \[
  \frac{dx}{dt} = \alpha x(1 - \frac{x}{K})
  \]
- The solution to the logistic growth equation is:
  \[
  x(t) = \frac{K0e^{\alpha t}}{K + \alpha(e^{\alpha t} - 1)} = \frac{K}{1 - \frac{K(1-K)}{K0e^{\alpha t}}}
  \]
  where \( \lim_{t \to \infty} x(t) = K \)
The Rosenzweig and MacArthur model

To address the second problem, the RM model assumes:

- The rate of an individual predator consumes prey has a maximum value:
  \[ \frac{\delta x}{b + x} \Rightarrow \delta \frac{1}{b + 1} \]

- When \( x \) is getting very large:
  \[ \lim_{x \to \infty} \delta \frac{1}{b + 1} = \delta \]

- The parameter \( b \) controls how rapidly an individual predator becomes full

Definition: Let’s put them together

The equations of RM model:

\[
\begin{align*}
\frac{dx}{dt} &= x(\alpha(1 - \frac{x}{K}) - \beta \frac{y}{b + x}) \\
\frac{dy}{dt} &= -y(\gamma - \delta \frac{x}{b + x})
\end{align*}
\]

The prey population grows according to the logistic growth equation with a carrying capacity of the prey population \( K \). The rate of an individual predator consume prey is controlled by parameter \( b \) and has a maximum value of \( \delta \).

The RM model in MATLAB

Step 1: Create a MATLAB function that defines the rate of change of the vector \( y \)

```matlab
function dy = R_M_Model(t,y)
% Rosenzweig and MacArthur predator-prey model.
dy = zeros(2,1);
alpha = 0.9; beta = 1;
gamma = 0.2; delta = 0.3;
K = 100; b=10;

dy(1) = (alpha.*((1-y(1))./K) - ... 
    beta.*y(2)./(b+y(1))).*y(1);

dy(2) = (gamma - delta.*y(1)./(b+y(1))) *y(2);
end
```

Step 2: Call this function using a numerical solver `ode45`

```matlab
1 t0 = 0;
2 tfinal = 300;
3 y0 = [10; 1];
4 % Simulate the differential equation.
5 [t,y] = ode45('R_M_Model',[t0 tfinal],y0);
```

Simulation: Let’s run the model

Satisfied?

Mostly. We will analyse the results in our next lecture.