A Short Note on Intuitionistic Propositional Logic with Multiple Conclusions

Valeria de Paiva
ISL, PARC, USA

Luiz Carlos Pereira
Philosophy, PUC-Rio, Brazil

Abstract

It is a common misconception among logicians to think that intuitionism is necessarily tied-up with single conclusion (sequent or Natural Deduction) calculi. Single conclusion calculi can be used and are convenient, but they are by no means necessary, as shown by such influential authors as Kleene, Takeuti and Dummett, to cite only three. If single conclusions are not necessary, how do we guarantee that only intuitionistic derivations are allowed? Traditionally one insists on restrictions on particular rules: implication right, negation right and universal quantification right are required to be single conclusion rules. In this note we show that instead of a cardinality restrictionm such as one conclusion only, we can use a notion of *dependency* between formulae to enforce the constructive character of derivations.

Since Gentzen's pioneering work it has been traditional to associate intuitionism with a single-conclusion sequent calculus or Natural Deduction system. Gentzen's own sequent calculus presentation of intuitionistic logic, the famous system LJ, is obtained from his classical system LK by means of a cardinality restriction imposed on the succedent of every sequent. It is well-known that Gentzen's formulation of classical logic, the system LK, uses sequents, expressions of the form $\Gamma \Rightarrow \Delta$, where both Γ and Δ may contain several formula occurrences. The intuition is that the conjunction of the formulae in Γ entails the disjunction of the formulae in Δ. In Gentzen's calculus for intuitionistic logic LJ sequents are restricted to succedents with *at most* one formula occurrence. This is convenient, but by no means necessary.

Since at least Maehara's work in the fifties (see [Mae54]) it has been known that intuitionistic logic can be presented via a multiple-conclusion system. Maehara's system is described in Takeuti's influential book ([Tak75]), which calls the system LJ'. Also Kleene in his monograph ([Kl52]) presents systems which constitute multiple-conclusion versions of intuitionistic logic. But while both of these (classes of) systems stick to the idea that sequents can have multiple conclusions, they still keep some form of (local) cardinality restriction on succedents: the rules for implication right, negation right and universal quantification right must be modified in that they can only be performed if there is a single formula in the succedent of the premise to which these rules are applied. If we do not impose these (local) restrictions, the systems collapse back into classical logic.

In the nineties the authors devised a sequent calculus for (propositional) intuitionistic logic, the system FIL [dPP95], where all rules may have multiple succedents, just as they do in classical logic. To make sure that classical inferences would not go through, we considered a notion of *dependency* between formulae and added a side-condition to the implication right rule based on these dependencies. Negation was treated as implication into absurdity, as usual in intuitionistic logic. In this way, cardinality restrictions (local or global) were replaced by restrictions based on control mechanisms over dependency relations.

The original motivation for the work on FIL stemmed from Linear Logic. Valeria de Paiva and Martin Hyland were at that time working with full intuitionistic linear logic (FILL)[HdP93], that is, linear logic with the full class of additive and multiplicative operators, and in order to obtain sequent rules for an intuitionistic multiplicative disjunction, they needed a multiple conclusions system. Their first attempt was to use Maehara's LJ', but this attempt was soon abandoned after the discovery of counter examples[1]cut-elimination ([Sch91]). Different FILL systems based on multiple conclusions and dependency relations exist in the literature (see [BdP97],[Bie96]).

The idea of using dependency relations to describe intuitionistic logic, as opposed to linear logic, was suggested to the authors by Martin Hyland. While working with FIL, Pereira realized that the intuitionistic system FIL could be used to solve the (old) problem of finding a cut-free system for the logic of constant domains, one of the best known intermediary logics.The logic of constant domains is an extension of intuitionistic first order logic (an intermediary logic) obtained by the addition of the axiom scheme $(\forall x(A \vee B(x)) \rightarrow (A \vee \forall x B(x))$, with x not free in A. The authors discovered afterwards that a FIL-like system had been independently worked on and published by Ryo Kashima[Kas91] and Kashima and Shimura[KS94]. Kashisma and Shimura provided the first proof of the cut-elimination theorem for the logic of constant domains, but they did not describe their system as based on a multiple succedent system for intuitionistic propositional logic itself, thereby alienating any readers not interested in intermediary logics.

Another nice application of the basic intuition of FIL was found by Torben Braüner, who presented (as far as we know, for the first time) a cut-free system for the modal logic S5. A Natural Deduction version of FIL, the system NFIL, was defined and studied by Ludmilla Franklin, who proved the equivalence between FIL and NFIL, and the normalization theorem for NFIL, in her master thesis[Fra00] written under Pereira's supervision.

[1] A counter example to cut-elimination in FILL due to Pereira is given by the derivation:

$$\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow \bot, p} \qquad \dfrac{\bot, 0 \Rightarrow q}{\bot \Rightarrow 0 \multimap q}}{p \Rightarrow (0 \multimap q), p}$$

The remaining part of this note is organized as follows. We will give a rough description of the system FIL and its cut-elimination strategy. We then show how to obtain the two applications we mentioned above, to wit, the logic of constant domains and full intuitionistic linear logic. In the final part of the note, we discuss extensions of the basic intuitions on FIL and future work.

1 Full Intuitionistic Logic

In this section we present the sequent calculus FIL for Full Intuitionistic Logic. FIL is a multiple succedent intuitionistic system, where an indexing device allows us to keep track of dependency relations between formulas in the antecedent and in the succedent of a sequent. Dependency relations determine the restriction in the formulation of the rule for introduction of implication on the right ($\Rightarrow\rightarrow$) that guarantees that only intuitionistic valid formulas are derived.

First we introduce some conventions we will be using throughout. Formulas (ranged over by A, B) are built in the usual way from propositional variables, propositional connectives for conjunction $\wedge$, disjunction $\vee$ and implication $\rightarrow$, and the constant $\perp$ for absurdity. As usual, negation $\neg A$ is defined as $(A \rightarrow \perp)$.

Definition 1. *A (decorated) sequent is an expression of the form*

$$A_1(n_1), \ldots, A_k(n_k) \Rightarrow B_1/S_1, \ldots, B_m/Sm$$

where:

- A_i *for* $(1 \leq i \leq k)$ *and* B_j *for* $(1 \leq j \leq m)$ *are formulae;*
- n_i *for each* i *such that* $(1 \leq i \leq k)$ *is a natural numbers and for all* i, j $(1 \leq i, j \leq k)$, $n_i \neq n_j$. *We say that* n_i *is the* index *of the formula* A_i*;*
- S_j *for each* j, $(1 \leq j \leq m)$ *is a set of natural numbers. We call* S_j *the* dependency set *of the formula* B_j.

Our decorated sequents have an index n_i for each formula in the antecedent A_i and a dependency set S_j for each formula in the consequent B_j. The main intuition is that the set of natural numbers S_j records which formulae in the antecedent the succedent formula B_j depends on. This extended notion of sequent can also be seen as a simplification of a term assignment judgement. Capital Greek letters like Γ and Δ denote sequences of indexed formulas either in the antecedent or in the succedent.

To describe the inference rules of our sequent calculus we need some notational conventions. Assume that the set of formulas Δ consists of indexed formulae and dependency sets $B_1/S_1, \ldots, B_m/S_m$,

1. If S is any set of natural numbers, $\Delta[k|S]$ denotes the result of replacing each S_j in Δ such that $k \in S_j$ by $(S_j \setminus \{k\}) \cup S$;
2. $\Delta \setminus \{n\}$ is the result of replacing each S in Δ such that $n \in S$ by $S \setminus \{n\}$.

3. $\Delta[k, S']$ denotes the result of replacing each S in Δ such that $k \in S$ by $S \cup S'$.

The system FIL is given by the axioms and rules of inference in Figure 1, together with the following conventions.

- We assume that in the case of the rules for conjunction on the right ($\Rightarrow \wedge$), disjunction on the left ($\vee \Rightarrow$), implication on the left ($\rightarrow\Rightarrow$) and Cut, the upper sequents of the premises have no index in common. This is in fact no strong restriction, since we can always rename the indices.
- In the *Cut* rule $\Delta'^{*} = \Delta'[n|S]$.
- In the rule for weakening on the left ($Weak_{\mathcal{L}}$), n is a new index and Δ^{*} is obtained from Δ through the introduction of n in at least one set S of dependencies in Δ.
- In the rule for contraction on the left, ($Contr_{\mathcal{L}}$), the new index k is the minimum $min(n, m)$ of the contracted indices n and m and Δ^{*} is obtained as $\Delta[(max(n, m), k]$.
- In the rule for disjunction on the right ($\vee_{\mathcal{R}}$), k is a new index and in the succedent of the conclusion $\Delta^{*} = \Delta[n, k]$ and $\Delta'^{*} = \Delta'[n, k$.
- In the rule for conjunction on the left ($\wedge_{\mathcal{L}}$), k is the minimum index between n and m and in $\Delta^{*} = \Delta[max(n, m), k]$.
- In the rule for implication on the left ($\rightarrow_{\mathcal{L}}$), $\Delta'^{*} = \Delta'[n, S]$.
- Finally, and most importantly, in the rule for implication on the right ($\rightarrow_{\mathcal{R}}$), n is an index in S and for every S' in Δ, $n \notin S'$. If this restriction is satisfied, it means that no other formula in Δ depends on the indicated occurrence of A.

Let us illustrate how the system works with two examples of derivations.

Example 1.

$$\cfrac{\cfrac{\cfrac{A(1) \Rightarrow A/\{1\}}{B(2), A(1) \Rightarrow A/\{1, 2\}}}{A(1) \Rightarrow (B \rightarrow A)/\{1\}}}{\Rightarrow (A \rightarrow (B \rightarrow A)/\{\}}$$

$$\frac{}{A(n) \Rightarrow A/\{n\}}\ Axiom$$

$$\frac{\Gamma \Rightarrow A/S, \Delta \qquad A(n), \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'^{*}}\ Cut$$

$$\frac{}{\bot(n) \Rightarrow A_1/\{n\}, \ldots, A_k/\{n\}}\ (\bot_{\mathcal{L}})$$

$$\frac{\Gamma, A(n), B(m), \Gamma' \Rightarrow \Delta}{\Gamma, B(m), A(n), \Gamma' \Rightarrow \Delta}\ (Exch_{\mathcal{L}}) \qquad \frac{\Gamma \Rightarrow A/S, B/S', \Delta}{\Gamma \Rightarrow B/S', A/S, \Delta}\ (Exch_{\mathcal{R}})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A(n) \Rightarrow \Delta^{*}}\ (Weak_{\mathcal{L}}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A/\{\}, \Delta}\ (Weak_{\mathcal{R}})$$

$$\frac{\Gamma, A(n), A(m) \Rightarrow \Delta}{\Gamma, A(k) \Rightarrow \Delta^{*}}\ (Contr_{\mathcal{L}}) \qquad \frac{\Gamma \Rightarrow A/S, A/S', \Delta}{\Gamma \Rightarrow A/S \cup S', \Delta}\ (Contr_{\mathcal{R}})$$

$$\frac{\Gamma, A(n) \Rightarrow \Delta \quad \Gamma', B(m) \Rightarrow \Delta'}{\Gamma, \Gamma', (A \lor B)(k) \Rightarrow \Delta^{*}, \Delta'^{*}}\ (\lor_{\mathcal{L}}) \qquad \frac{\Gamma \Rightarrow A/S, B/S', \Delta}{\Gamma \Rightarrow (A \lor B)/S \cup S', \Delta}\ (\lor_{\mathcal{R}})$$

$$\frac{\Gamma, A(n), B(m) \Rightarrow \Delta}{\Gamma, (A \land B)(k) \Rightarrow \Delta}\ (\land_{\mathcal{L}}) \qquad \frac{\Gamma \Rightarrow A/S, \Delta^{*} \qquad \Gamma' \Rightarrow B/S', \Delta'}{\Gamma, \Gamma' \Rightarrow (A \land B)/S \cup S', \Delta, \Delta'}\ (\land_{\mathcal{R}})$$

$$\frac{\Gamma \Rightarrow A/S, \Delta \qquad \Gamma', B(n) \Rightarrow \Delta'}{\Gamma, \Gamma', (A \to B)(n) \Rightarrow \Delta, \Delta'^{*}}\ (\to_{\mathcal{L}}) \qquad \frac{\Gamma, A(n) \Rightarrow B/S, \Delta}{\Gamma \Rightarrow (A \to B)/S - \{n\}, \Delta}\ n \text{ not in } (\to_{\mathcal{R}})$$

Figure 1: Sequent Calculus formulation of FIL

Example 2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(1) \Rightarrow A/\{1\} \quad B(2) \Rightarrow B/\{2\}}{(A \lor B)(3) \Rightarrow A/\{3\}, B/\{3\}} \qquad C(4) \Rightarrow C/\{4\}}{(A \lor B)(3), (B \to C)(4) \Rightarrow A/\{3\}, C/\{3,4\}}}{(A \lor B)(3), (B \to C)(4) \Rightarrow (A \lor C)/\{3\}, C/\{3,4\}}}{(A \lor B)(3), (B \to C)(4) \Rightarrow (A \lor C)/\{3\}, (A \lor C)/\{3,4\}}}{(A \lor B)(3), (B \to C)(4) \Rightarrow (A \lor C)/\{3,4\}}}{(A \lor B)(3) \Rightarrow [(B \to C) \to (A \lor C)]/\{3\}}$$

Consider now the following attempt at a derivation of the law of the excluded middle:

$$\cfrac{\cfrac{A(1) \Rightarrow A/\{1\}}{A(1) \Rightarrow A/\{1\}, \bot/\{\}}}{\Rightarrow A/\{1\}, (A \rightarrow \bot)/??}$$

The last inference cannot be made as neither the index 1 is in the empty set nor the index 1 is not in $\{1\}$. So the side-condition for implication is not satisfied, showing that this simple proof of the excluded middle does not go through.

The system FIL is sound and complete with respect to LJ. The proof of completeness is completely straightforward: every proof in LJ can be easily "decorated" with labels and transformed into a proof in FIL. The proof of soundeness is more involved and can be obtained, for example, from the proof of Kashima and Shimura (for the case of the logic of constant domains) if we leave out of the proof the first-order apparatus.

The proof of the cut-elimination theorem for FIL is quite standard. As usual, the rule of contraction presents some difficulties that are dealt with through the use and eliminability of a generalized form of cut; instead of eliminating simple cuts, we show how to eliminate indexed cuts (see [Sch91]). An indexed cut is defined as follows:

$$\frac{\Gamma \Rightarrow \Delta \qquad \Theta \Rightarrow \Lambda}{\Gamma, \Theta' \Rightarrow \Delta', \Lambda}\,(A, n_1, \ldots, n_k; m_1, \ldots, m_j)$$

The information standing on the right of the inference line indicates that the cut-formula is A, and that $\Delta'(\Theta')$ is obtained through the deletion of the cut-formula A in positions $n_1, \ldots, n_k(m_1, ..., m_j)$ in $\Delta(\Theta)$.

It is routine matter to prove that the indexed-cut rule is equivalent to the simple cut rule. Trivially, an application of the cut rule is an (unary) application of indexed-cut. The other side of the equivalence can be proved with the use of permutations, contractions and the cut-rule. Details are left to the reader. Similar to Gentzen's original proof of the *Hauptsatz* we prove the following basic lemma:

Lemma 2. *Let* Π *be a derivation of* $\Gamma \Rightarrow \Delta$ *in* FIL *such that:*

1. *The last rule applied in* Π *is an indexed-cut;*
2. *There is no other application of indexed-cut in* Π.

Then Π *can be transformed into an indexed cut-free derivation* Π' *of* $\Gamma \Rightarrow \Delta$.

Proof: By a routine induction over (lexicographically ordered) pairs (α, β), where α is the degree of the cut-formula and β its rank, as in Gentzen's original proof. In fact, we are not really working with Gentzen's *rank* but rather with the sum of the longest *cluster sequences* for the cut-formulas.

We show one case where the left-rank is equal to 1, and the right-rank is greater than 1, the right upper sequent being obtained by an application of the implication-right rule.

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A/S}{\Gamma \Rightarrow \Delta, A'/S'} \qquad \dfrac{\Gamma_1, C(n) \Rightarrow \Delta_1, D/S''}{\Gamma_1 \Rightarrow \Delta_1, (C \to D)/S'' - \{n\}}}{\Gamma, \Gamma_1^* \Rightarrow \Delta, \Delta_1^*, (C \to D)(S'' - \{n\})^*}$$

This derivation can be transformed into

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A/S}{\Gamma \Rightarrow \Delta, A'/S'} \qquad \Gamma_1, C(n) \Rightarrow \Delta_1, D/S''}{\Gamma, \Gamma_1^* \Rightarrow \Delta, \Delta_1^*, (C \to D)/(S'')^* - \{n\}}$$

The result now follows directly from the side-condition on the implication rule, as n is not an element of S', and $(S'')^* - \{n\} = (S'' - \{n\})^*$

Theorem 3 (Cut-elimination). *If* Π *is a derivation of* $\Gamma \Rightarrow \Delta$ *in FIL, then* Π *can be transformed into a cut-free proof* Π' *of* $\Gamma \to \Delta$.

Proof: Directly from the basic lemma above.

2 Two Applications of FIL

The two original applications of the system FIL were related to Linear Logic and to the Logic of Constant Domains. The system FIL was introduced in connection with a variant of Linear Logic, called Full Intuitionistic Linear Logic (FILL)[HdP93]. On the one hand, the intuitionistic nature of FILL required that its logical operators were not defined in terms of each other, and hence one could not use the duality to define the multiplicative disjunction, the connective par, written as $\parr$, in terms of tensor, say. On the other hand, the multiplicative disjunction demanded a multiple succedent in the formulation of the rule of disjunction on the right: the condition for introducing $(A \parr B)$ is that we have "disjunctively" both A and B, the classical rule for disjunction being

$$\dfrac{\Gamma \Rightarrow \Delta, A/S, B/S'}{\Gamma \Rightarrow \Delta, (A \parr B)/S \cup S'}$$

The cut-elimination theorem holds for the linear system FILL and a nice proof of it can be found in [BdP97]. Formulations of Full Intuitionistic Linear Logic using proof-nets can be found in [Bel97] and using patterns in [Bie96].

In the early eighties it was conjectured that the Logic of Constant Domains (CD) would not admit a "complete, cut-free, sequent axiomatization" (see [L-E83]), where the cut-free axiomatizations should satisfy the following two criteria:

1. The axioms should contain only atomic formulae; and
2. Each operational rule should introduce a single logical operator.

The problem of finding a cut-free system for CD was solved by Ryo Kashima and Tatsuya Shimura in 1994 (see [Kas91] and Kashima and Shimura) through the use of a FIL-like system CLD. The main idea for the definition of CLD was to use a binary relation, called a "connection", expressing "a dependency between formulas in the antecedent and formulas in the succedent of a sequent". In order to obtain a cut-free sequent calculus for the Logic of Constant Domains we simply add the first order rules to FIL, with dependencies being inherited in the quantifier rules, as in the example below:

$$\frac{A(t)(n), \Gamma \Rightarrow \Delta}{\forall x.A(x)(n), \Gamma \Rightarrow \Delta}\ (\forall_{\mathcal{R}})$$

The system CLD, as published, contains a FIL-like basic system, but its authors never made the point that this implicit system could be used as an alternative way of formulating intuitionistic propositional logic.

3 Concluding Remarks

This short note described the multiple conclusions system FIL for propositional intuitionistic logic and recounted some of its original applications. We are convinced that FIL is not simply a trick that logicians can play to reformulate a traditional sequent system. Dependency relations and discharging functions play a fundamental role in the proof theory of natural deduction systems. Crude discharging strategies for several well-known natural deduction systems entail that good proof-theoretical properties are lost (e.g. strong-normalization, uniqueness of normal form, generality). From a very general conceptual point of view, FIL shows that classical logic can be distinguished from intuitionistic logic by the way dependency relations and discharging functions are handled in both systems. The introduction of multiple conclusions in intuitionistic logic imposes some constraints on which assumptions are available for discharge at each conclusion. In the case of propositional logic this means that in a multiple conclusion system, it is the rule for implication introduction that distinguishes classical logic from intuitionistic logic: every assumption in classical logic is global, while some assumptions in intuitionistic logic may be local.

The idea that the distinction between intuitionistic logic and classical lies in the rule for implication introduction also suggests that there might be interesting translations from classical logic into intuitionistic logic which are not based on negation, but rather on implication. We plan to investigate these prospective translations in future work.

As for other future work, the system FIL is a propositional system and we know that the simply addition of the first order apparatus to FIL does not produce Intuitionistic First order Logic, but rather the Logic of Constant Domains.

Quantifier rules require a different kind of dependency relation (connection) that has not so far been defined. The same applies to some intuitionistic modal logics: if we add the modal apparatus of S4 to FIL, we obtain a kind of ”intermediary” modal logic S4, lying between classical S4 and (one of the several proposals for) intuitionistic S4. The proof theory and the model theory of this intermediary S4 have been studied ([Med01]), but no multiple conclusion (strictly) intuitionistic S4 has so far been defined.

A natural deduction version NFIL of the system FIL has already been defined, see [Fra00]. But we do not have yet a more "liberal" version of semantic tableaux that incorporates the idea of dependency relations into its rules. Traditional intuitionistic tableaux have the strong restriction that on each branch there can be at most one formula f. The use of dependency relations might produce more flexible tableaux, where multiple f-formulas are allowed to occur on the same branch.

References

[Bel97] G. Bellin. Subnets of Proofnets in Mutiplicative Linear Logic with MIX . In *Mathematical Structures in Computer Science*, vol 7, pp 663-699, 1997.

[Bie96] G. Bierman. A note on Full Intuitionistic Linear Logic. In *Annals of Pure and Applied Logic*, 79(3):282–287, 1996.

[Bra00] T. Braüner. A Cut-free Gentzen Formulation of the Modal Logic S5. In *Journal of the IGPl*, vol 8, 2000.

[BdP97] T. Braüner and V. de Paiva. A Formulation of Linear Logic Based on Dependency-Relations. In *Proc. of the Association of European Computer Science Logic*, Springer LNCS vol 1414, 1997.

[Dum00] M. A. E. Dummett. Elements of Intuitionism. *OUP*, 1977.

[Fra00] L. S. Franklin. Sistemas com multiplas conclusoes para a logica proposicional intuitionista. *M Sc Dissertation*, PUC-Rio, 2000.

[HdP93] J. M. E. Hyland and V. de Paiva. Full Intuitionistic Linear Logic. In *Annals of Pure and Applied Logic*, 64(3), pp. 273–291, 1993.

[Kas91] R. Kashima. Cut-Elimination for the intermediate logic CD. Research Report on Information Sciences, C100, Tokyo Institute of Technology, 1991.

[KS94] R. Kashima and T. Shimura. Cut-Elimination for the Logic of Constant Domains In *Mathematical Logic Quarterly*, vol 40, pp 153–172.

[Kl52] S. C. Kleene. Introduction to Metamathematics. *North-Holland*, Amsterdam, 1952.

[L-E83] E. G. K. Lopez-Escobar. A second paper on "The interpolation Theorem for the Logic of Constant Domains". In *The Journal of Symbolic Logic*, vol 48, pp. 595-599 1983.

[Mae54] S. Maehara Eine Darstellung der intuitionistischen Logik in der klassischen In *Nagoya Mathematical Journal*, 7, pp. 45 -64, 1954.

[Med01] M. da P. N.de Medeiros. Investigacoes acerca de uma versao modal para a logica intermediaria dosdominios constantes. In *Principios*, vol 8, no. 19, 2001.

[dPP95] V. de Paiva and L. C. Pereira. A New Proof System for Intuitionistic Logic. In *The Bulletin of Symbolic Logic*, 1(1), p. 101, 1995 (abstract).

[Pra65] D. Prawitz. *Natural Deduction: A Proof-Theoretic Study.* Almqvist and Wiksell, 1965.

[Sch91] H. Schellinx. Some syntactical observations in Linear Logic. In *Journal of Logic and Computation*, 1(4), 1991

[Tak75] G. Takeuti Proof Theory, North-Holland, 1975.