FO = FO$^3$ for Linear Orders with Monotone Binary Relations

Marie Fortin

University of Liverpool

YR-OWLS, June 16, 2020
The $k$-variable property

How many variables are needed in first-order logic?
The $k$-variable property

How many variables are needed in first-order logic?

- Some properties require unboundedly many variables

$$\exists x_1. \exists x_2. \exists x_3. \exists x_4. \quad \land_{1 \leq i < j \leq 4} x_i \neq x_j$$

Over linear orders, $\mathsf{FO}_2 = \mathsf{FO}_3$. 
The \( k \)-variable property

How many variables are needed in first-order logic?

- Some properties require unboundedly many variables

\[
\exists x_1. \exists x_2. \exists x_3. \exists x_4. \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j
\]

- ... but not in every class of models:
The $k$-variable property

How many variables are needed in first-order logic?

- Some properties require unboundedly many variables

\[ \exists x_1. \exists x_2. \exists x_3. \exists x_4. \land_{1 \leq i < j \leq 4} x_i \neq x_j \]

- ... but not in every class of models:

\[ \exists x. \exists y. (x < y \land \exists x. (y < x \land \exists y. x < y)) \]
The $k$-variable property

How many variables are needed in first-order logic?

- Some properties require unboundedly many variables

$$\exists x_1. \exists x_2. \exists x_3. \exists x_4. \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$$

- ... but not in every class of models:

$$\exists x. \exists y. (x < y \land \exists x. (y < x \land \exists y. x < y))$$
The $k$-variable property

How many variables are needed in first-order logic?

- Some properties require unboundedly many variables

$$\exists x_1. \exists x_2. \exists x_3. \exists x_4. \land_{1 \leq i < j \leq 4} x_i \neq x_j$$

- ... but not in every class of models:

$$\exists x. \exists y. (x < y \land \exists x. (y < x \land \exists y. x < y))$$

Over linear orders, $\text{FO} = \text{FO}_3$. 
The $k$-variable property

How many variables are needed in first-order logic?

- Some properties require unboundedly many variables

$$\exists x_1. \exists x_2. \exists x_3. \exists x_4. \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$$

- ... but not in every class of models:

$$\exists x. \exists y. (x < y \land \exists x. (y < x \land \exists y. x < y))$$
The \( k \)-variable property

How many variables are needed in first-order logic?

- Some properties require unboundedly many variables

\[ \exists x_1. \exists x_2. \exists x_3. \exists x_4. \land_{1 \leq i < j \leq 4} x_i \neq x_j \]

- ... but not in every class of models:

\[ \exists x. \exists y. (x < y \land \exists x. (y < x \land \exists y. x < y)) \]
The $k$-variable property

How many variables are needed in first-order logic?

- Some properties require unboundedly many variables

$$\exists x_1 . \exists x_2 . \exists x_3 . \exists x_4 . \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$$

- ... but not in every class of models:

$$\exists x . \exists y . (x < y \land \exists x . (y < x \land \exists y . x < y))$$

Over linear orders, $\text{FO} = \text{FO}^3$. 
Bounded variable logics

Why do we care about the number of variables?
Bounded variable logics

Why do we care about the number of variables?

- (Descriptive) complexity
Bounded variable logics

Why do we care about the number of variables?

- (Descriptive) complexity
- Temporal logics
Bounded variable logics

Why do we care about the number of variables?

- (Descriptive) complexity
- Temporal logics

[Gabbay 1981] In any class of time flows, TFAE:

- There exists an expressively complete finite set of FO-definable (multi-dimensional) temporal connectives
- There exists $k$ such that every first-order sentence is equivalent to one with at most $k$ variables
Example

Over linear orders, \( \text{FO} = \text{FO}^3 \).
Example

Over **linear orders**, $\text{FO} = \text{FO}^3$.

Two classical techniques to prove $\text{FO} = \text{FO}^k$ (over a class $C$)
Example

Over **linear orders**, $\text{FO} = \text{FO}^3$.

Two classical techniques to prove $\text{FO} = \text{FO}^k$ (over a class $C$)

1. Corollary of expressive completeness of a temporal logic
Example

Over **linear orders**, $\text{FO} = \text{FO}^3$.

Two classical techniques to prove $\text{FO} = \text{FO}^k$ (over a class $C$)

1. Corollary of expressive completeness of a temporal logic
   
   **Example:** Over complete linear orders,
   
   $\text{FO}^3 \subseteq \text{FO} = \text{LTL} \subseteq \text{FO}^3$  
   
   [Kamp 1968]
Two classical techniques to prove $\text{FO} = \text{FO}^k$ (over a class $C$)

1. Corollary of expressive completeness of a temporal logic

**Example:** Over complete linear orders,

$$\text{FO}^3 \subseteq \text{FO} = \text{LTL} \subseteq \text{FO}^3$$

[Kamp 1968]

Over (arbitrary) linear orders,

$$\text{FO}^3 \subseteq \text{FO} = \text{LTL with Stavi connectives} \subseteq \text{FO}^3$$

[Gabbay, Hodkinson, Reynolds 1993]
Example

**Over linear orders,** \( \text{FO} = \text{FO}^3 \).

Two classical techniques to prove \( \text{FO} = \text{FO}^k \) (over a class \( C \))

1. Corollary of expressive completeness of a temporal logic
2. Ehrenfeucht-Fraïssé games with \( k \) pebbles
Over linear orders, $\text{FO} = \text{FO}^3$.

Two classical techniques to prove $\text{FO} = \text{FO}^k$ (over a class $\mathcal{C}$)

1. Corollary of expressive completeness of a temporal logic
2. Ehrenfeucht-Fraïssé games with $k$ pebbles

Example: Over complete linear orders,
$$\text{FO} = \text{FO}^3$$
[Immerman, Kozen 1989]
Example

Over **linear orders**, $\text{FO} = \text{FO}^3$.

Two classical techniques to prove $\text{FO} = \text{FO}^k$ (over a class $C$)

1. Corollary of expressive completeness of a temporal logic
   0 or 1 free variables

2. Ehrenfeucht-Fraïssé games with $k$ pebbles
   up to $k$ free variables
Known results (non-exhaustive)

Over linear orders, $\forall k, \text{FO} \neq \text{FO}^k$ [Rossman'08]

Over $(\mathbb{R}, <, +1)$, $\text{FO} = \text{FO}^3$ [AHRW'15]

Over Mazurkiewicz traces, $\text{FO} = \text{FO}^3$ [Gastin-Mukund'02]

Over MSCs, $\text{FO} = \text{FO}^3$ [Bollig-F.-Gastin'18]

What do these 4 positive results have in common?
Known results (non-exhaustive)

Over linear orders,
\[ \text{FO} = \text{FO}^3 \]
[Immerman-Kozen’89]

What happens if we have additional binary relations?
Known results (non-exhaustive)

Over linear orders,
\[ \text{FO} = \text{FO}^3 \]
[Immerman-Kozen’89]

What happens if we have additional binary relations?

Over ordered graphs,
\[ \forall k, \text{FO} \neq \text{FO}^k \]
[Rossman’08]

Over \((R, <, +1)\),
\[ \text{FO} = \text{FO}^3 \]
[AHRW’15]

Over Mazurkiewicz traces,
\[ \text{FO} = \text{FO}^3 \]
[Gastin-Mukund’02]

Over MSCs,
\[ \text{FO} = \text{FO}^3 \]
[Bollig-F.-Gastin’18]
Known results (non-exhaustive)

Over linear orders,
\[ \text{FO} = \text{FO}^3 \]
[Immerman-Kozen’89]

What happens if we have additional binary relations?

Over ordered graphs,
\[ \forall k, \text{FO} \neq \text{FO}^k \]
[Rossman’08]

Over \((\mathbb{R}, <, +1)\),
\[ \text{FO} = \text{FO}^3 \]
[AHRW’15]
Known results (non-exhaustive)

Over linear orders, $\text{FO} = \text{FO}^3$  
[Immerman-Kozen’89] ✓

What happens if we have additional binary relations?

Over ordered graphs, $\forall k, \text{FO} \neq \text{FO}^k$  
[Rossman’08] ×

Over $(\mathbb{R}, <, +1)$, $\text{FO} = \text{FO}^3$  
[AHRW’15] ✓

Over Mazurkiewicz traces, $\text{FO} = \text{FO}^3$  
[Gastin-Mukund’02] ✓

Over MSCs, $\text{FO} = \text{FO}^3$  
[Bollig-F.-Gastin’18] ✓
Known results (non-exhaustive)

Over linear orders,

\[ \text{FO} = \text{FO}^3 \]

[Immerman-Kozen’89]

What happens if we have additional binary relations?

Over ordered graphs,

\[ \forall k, \text{FO} \neq \text{FO}^k \]

[Rossman’08]

Over \((\mathbb{R}, <, +1)\),

\[ \text{FO} = \text{FO}^3 \]

[AHRW’15]

Over Mazurkiewicz traces,

\[ \text{FO} = \text{FO}^3 \]

[Gastin-Mukund’02]

Over MSCs,

\[ \text{FO} = \text{FO}^3 \]

[Bollig-F.-Gastin’18]

What do these 4 positive results have in common?
Generalisation [F.’19]

$\text{FO} = \text{FO}^3$ over structures with

- one linear order $\leq$,
- “interval-preserving” binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$

Diagram:

```
  p
  ▼   ▼
p, q  q    p  p  p, q  q
```

Generalisation [F.’19]

$\text{FO} = \text{FO}^3$ over structures with

- one linear order $\leq$,
- “interval-preserving” binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$

$R$ is interval-preserving if for all intervals $I$,

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\text{dom}(R), \leq)$
Generalisation [F.’19]

$\text{FO} = \text{FO}^3$ over structures with

- one linear order $\leq$,
- “interval-preserving” binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$

$R$ is interval-preserving if for all intervals $I$,

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\text{dom}(R), \leq)$
Generalisation [F.’19]

$\text{FO} = \text{FO}^3$ over structures with

- one linear order $\leq$,
- “interval-preserving” binary relations $R_1, R_2, \ldots$,
- arbitrary unary predicates $p, q, \ldots$

$R$ is interval-preserving if for all intervals $I$,

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\text{dom}(R), \leq)$
Generalisation \[F.'19\]

\[\text{FO} = \text{FO}^3 \text{ over structures with}\]
- one linear order \(\leq\),
- “interval-preserving” binary relations \(R_1, R_2, \ldots\),
- arbitrary unary predicates \(p, q, \ldots\)

\(R\) is interval-preserving if for all intervals \(I\),
- \(R(I)\) is an interval of \((\text{Im}(R), \leq)\)
- \(R^{-1}(I)\) is an interval of \((\text{dom}(R), \leq)\)
Generalisation [F.’19]

\[ \text{FO} = \text{FO}^3 \text{ over structures with} \]

- one linear order \( \leq \),
- “interval-preserving” binary relations \( R_1, R_2, \ldots \),
- arbitrary unary predicates \( p, q, \ldots \)

\[ R \text{ is interval-preserving if for all intervals } I, \]

- \( R(I) \) is an interval of \( (\text{Im}(R), \leq) \)
- \( R^{-1}(I) \) is an interval of \( (\text{dom}(R), \leq) \)
Generalisation [F.’19]

\( FO = FO^3 \) over structures with

- one linear order \( \leq \),
- “interval-preserving” binary relations \( R_1, R_2, \ldots \),
- arbitrary unary predicates \( p, q, \ldots \)

\( R \) is interval-preserving if for all intervals \( I \),

- \( R(I) \) is an interval of \( (\text{Im}(R), \leq) \)
- \( R^{-1}(I) \) is an interval of \( (\text{dom}(R), \leq) \)
A special case: monotone partial functions

Any relation $R$ corresponding to a monotone partial function is interval-preserving.
A special case: monotone partial functions

Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
A special case: monotone partial functions

Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$

![Diagram showing an interval $I$]
Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
A special case: monotone partial functions

Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
A special case: monotone partial functions

Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
A special case: monotone partial functions

Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\text{dom}(R), \leq)$
A special case: monotone partial functions

Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\text{dom}(R), \leq)$
A special case: monotone partial functions

Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\text{dom}(R), \leq)$
Any relation $R$ corresponding to a monotone partial function is interval-preserving.

- $R(I)$ is an interval of $(\text{Im}(R), \leq)$
- $R^{-1}(I)$ is an interval of $(\text{dom}(R), \leq)$
Applications

\[ FO = FO^3 \] over

1. Linear orders with partial non-decreasing or non-increasing functions (new)
Applications

$\text{FO} = \text{FO}^3$ over

1. Linear orders with partial non-decreasing or non-increasing functions \textit{(new)}

2. Linear orders: finite or infinite words, $\mathbb{R}$, $\mathbb{Q}$, ordinals...
Applications

$\text{FO} = \text{FO}^3$ over

1. Linear orders with partial non-decreasing or non-increasing functions (new)
2. Linear orders: finite or infinite words, $\mathbb{R}$, $\mathbb{Q}$, ordinals...
3. $(\mathbb{R}, \leq, +1)$, $(\mathbb{R}, \leq, (+q)_{q \in \mathbb{Q}})$ ...
Applications

$FO = FO^3$ over

1. Linear orders with partial non-decreasing or non-increasing functions (new)
2. Linear orders: finite or infinite words, $\mathbb{R}$, $\mathbb{Q}$, ordinals...
3. $(\mathbb{R}, \leq, +1)$, $(\mathbb{R}, \leq, (+q)_{q\in\mathbb{Q}})$ ...
4. $(\mathbb{R}, \leq)$ + polynomial functions (new)
Applications

5. Message sequence charts (MSCs)
Applications

5. Message sequence charts (MSCs)

\[
\begin{align*}
&\text{p} \quad a \rightarrow a \rightarrow c \rightarrow a \rightarrow a \rightarrow a \rightarrow a \rightarrow a \\
&\text{q} \quad a \rightarrow a \rightarrow a \rightarrow a \rightarrow a \rightarrow a \rightarrow a \rightarrow a \\
&\text{r} \quad a \rightarrow b \rightarrow b \rightarrow a \rightarrow a \rightarrow c \rightarrow a \rightarrow a \rightarrow a
\end{align*}
\]
Applications

5. Message sequence charts (MSCs)

\[ p \quad a \to a \to c \to a \to a \to a \to a \to a \to a \]

\[ q \quad a \to a \to a \to a \to a \to a \to a \to a \to a \to a \]

\[ r \quad a \to b \to b \to a \to a \to c \to a \to a \to a \to a \]

Executions of message-passing systems
Applications

5. Message sequence charts (MSCs)

Executions of **message-passing systems**

- Fixed, finite set of processes
5. Message sequence charts (MSCs)

Executions of **message-passing systems**

- Fixed, finite set of processes
- Process order $\leq_{\text{proc}}$
Applications

5. Message sequence charts (MSCs)

Executions of **message-passing systems**
- Fixed, finite set of processes
- Process order $\leq_{\text{proc}}$
- Message relations $\triangleleft_{p,q}$
Applications

5. Message sequence charts (MSCs)

 Executions of message-passing systems
  - Fixed, finite set of processes
  - Process order $\leq_{\text{proc}}$
  - Message relations $\triangleleft_{p,q}$
Applications

5. Message sequence charts (MSCs)

Executions of *message-passing systems*

- Fixed, finite set of processes
- Process order $\leq_{\text{proc}}$
- Message relations $\prec_{p,q}$

Extended to a linear order
5. Message sequence charts (MSCs)

Executions of **message-passing systems**
- Fixed, finite set of processes
- Process order $\leq_{\text{proc}}$
- Message relations $\prec_{p,q}$

Extended to a linear order
FIFO $\rightarrow$ monotone
Applications

5. Message sequence charts (MSCs)

Executions of **message-passing systems**

- Fixed, finite set of processes
- Process order $\leq_{\text{proc}}$
- Message relations $\prec_{p,q}$

Extended to a linear order

FIFO $\rightarrow$ monotone

$\rightarrow$ Interval-preserving structure
Applications

\( \text{FO} = \text{FO}^3 \) over structures with

- one linear order \( \leq \),
- “interval-preserving” binary relations \( R_1, R_2, \ldots \),
- arbitrary unary predicates \( p, q, \ldots \)

1. Linear orders with partial non-decreasing or non-increasing functions (\textit{new})
2. Linear orders: finite or infinite words, \( \mathbb{R}, \mathbb{Q}, \) ordinals...
3. \( (\mathbb{R}, \leq, +1), (\mathbb{R}, \leq, (+q)_{q \in \mathbb{Q}}) \) ...
4. \( (\mathbb{R}, \leq) + \) polynomial functions (\textit{new})
5. MSCs
6. Mazurkiewicz traces
How does the interval-preserving assumption help?
How does the interval-preserving assumption help?

\[ \varphi(x_1, x_2, x_3) = \exists y. R_1(x_1, y) \land R_2(x_2, y) \land R_3(x_3, y) \]
How does the interval-preserving assumption help?

\[ \varphi(x_1, x_2, x_3) = \exists y. R_1(x_1, y) \land R_2(x_2, y) \land R_3(x_3, y) \]

Equivalent FO\(^3\) formula?
How does the interval-preserving assumption help?

\[ \varphi(x_1, x_2, x_3) = \exists y. R_1(x_1, y) \land R_2(x_2, y) \land R_3(x_3, y) \]
How does the interval-preserving assumption help?

\[ \varphi(x_1, x_2, x_3) = \exists y. R_1(x_1, y) \land R_2(x_2, y) \land R_3(x_3, y) \]

Equivalent FO\(^3\) formula?
How does the interval-preserving assumption help?

\[ \varphi(x_1, x_2, x_3) = \exists y. R_1(x_1, y) \land R_2(x_2, y) \land R_3(x_3, y) \]

\[ \equiv \left( \exists y. R_1(x_1, y) \land R_2(x_2, y) \land \left( \exists y. R_1(x_1, y) \land R_3(x_3, y) \land \left( \exists y. R_2(x_2, y) \land R_2(x_3, y) \land \right) \right) \right) \land \]

Equivalent FO$^3$ formula?
How does the interval-preserving assumption help?

\[ \varphi(x_1, x_2, x_3) = \exists y. R_1(x_1, y) \land R_2(x_2, y) \land R_3(x_3, y) \]

\[ \equiv \left( \exists y. R_1(x_1, y) \land R_2(x_2, y) \land \exists x. R_3(x, y) \right) \land \]

\[ \left( \exists y. R_1(x_1, y) \land R_3(x_3, y) \land \exists x. R_2(x, y) \right) \land \]

\[ \left( \exists y. R_2(x_2, y) \land R_3(x_3, y) \land \exists x. R_1(x, y) \right) \]

Equivalent FO\(^3\) formula?
How does the interval-preserving assumption help?

\[ \varphi(x_1, x_2, x_3) = \exists y. R_1(x_1, y) \land R_2(x_2, y) \land R_3(x_3, y) \]

\[ \equiv \left( \exists x_3. R_1(x_1, x_3) \land R_2(x_2, x_3) \land \exists x_1. R_3(x_1, x_3) \right) \land \]
\[ \left( \exists x_2. R_1(x_1, x_2) \land R_3(x_3, x_2) \land \exists x_1. R_2(x_1, x_2) \right) \land \]
\[ \left( \exists x_1. R_2(x_2, x_1) \land R_2(x_3, x_1) \land \exists x_2. R_1(x_2, x_1) \right) \]

Equivalent FO³ formula?
The proof

\[ \text{FO} = \text{FO}^3 \text{ over structures with} \]

- one linear order \( \leq \),
- “interval-preserving” binary relations \( R_1, R_2, \ldots \),
- arbitrary unary predicates \( p, q, \ldots \)
The proof

**FO = FO³** over structures with
- one linear order ≤,
- “interval-preserving” binary relations $R_1, R_2, \ldots$, 
- arbitrary unary predicates $p, q, \ldots$

**Key idea:** Go through an intermediate language: Star-free Propositional Dynamic Logic.
Star-free Propositional Dynamic Logic

Examples
Star-free Propositional Dynamic Logic

Examples

\[ (p \land \neg q) \lor (q \land \neg p) \]

\[ \langle R \rangle q \]

\[ \langle \leq \cdot R \cdot \rangle q \]

\[ \langle \leq \cdot \{ \langle R \rangle q \} \rangle \]

\[ \langle R \cap \leq \rangle (p \land q) \]
Star-free Propositional Dynamic Logic

Examples

\[(p \land \neg q) \lor (q \land \neg p)\]
Star-free Propositional Dynamic Logic

Examples

\((p \land \neg q) \lor (q \land \neg p)\)  
\(\langle R \rangle q\)
Star-free Propositional Dynamic Logic

Examples

\[
(p \land \neg q) \lor (q \land \neg p)
\]

\[
\langle R \rangle q
\]

\[
\langle \leq \cdot R^{-1} \rangle q
\]

<table>
<thead>
<tr>
<th>Expression</th>
<th>p</th>
<th>q</th>
<th>q</th>
<th>p</th>
<th>p</th>
<th>p, q</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p \land \neg q) \lor (q \land \neg p)</td>
<td>x</td>
<td>✓</td>
<td>x</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>x</td>
</tr>
<tr>
<td>\langle R \rangle q</td>
<td>✓</td>
<td>x</td>
<td>✓</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>\langle \leq \cdot R^{-1} \rangle q</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>x</td>
</tr>
</tbody>
</table>
Star-free Propositional Dynamic Logic

Examples

\[
(p \land \neg q) \lor (q \land \neg p)
\]

\[
\langle R \rangle q
\]

\[
\langle \leq \cdot R^{-1} \rangle q
\]

\[
\langle \leq \cdot \{\langle R \rangle q? \cdot \leq \rangle p
\]

- \( R \)
- \( p \)
- \( q \)
- \( q \)
- \( p \)
- \( p \)
- \( p, q \)
- \( q \)
Star-free Propositional Dynamic Logic

Examples

\[(p \land \neg q) \lor (q \land \neg p)\]  
\[\langle R \rangle q\]  
\[\langle \leq \cdot R^{-1} \rangle q\]  
\[\langle \leq \cdot \{\langle R \rangle q\}? \cdot \leq \rangle p\]  
\[\langle R^c \cap \leq \rangle (p \land q)\]

<table>
<thead>
<tr>
<th>Expression</th>
<th>p</th>
<th>q</th>
<th>q</th>
<th>p</th>
<th>p</th>
<th>p, q</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>((p \land \neg q) \lor (q \land \neg p))</td>
<td>X</td>
<td>✓</td>
<td>X</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>(\langle R \rangle q)</td>
<td>✓</td>
<td>X</td>
<td>✓</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>(\langle \leq \cdot R^{-1} \rangle q)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>X</td>
</tr>
<tr>
<td>(\langle \leq \cdot {\langle R \rangle q}? \cdot \leq \rangle p)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>(\langle R^c \cap \leq \rangle (p \land q))</td>
<td>X</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>X</td>
</tr>
</tbody>
</table>
Star-free Propositional Dynamic Logic

Examples

Over \((\mathbb{R}, <, \{+q \mid q \in \mathbb{Q}_+\})\),

\[ \varphi \mathcal{U}_{(q,r)} \psi \equiv \]

\begin{align*}
\varphi & \quad t \\
\varphi & \quad t + q \\
\psi & \quad t + r
\end{align*}
Star-free Propositional Dynamic Logic

Examples

Over \((\mathbb{R}, <, \{ +q \mid q \in \mathbb{Q}_+ \})\),

\[ \varphi \mathcal{U}_{(q,r)} \psi \equiv \langle (+q \cdot <) \cap (+r \cdot <^{-1}) \cap (< \cdot \{\neg \varphi\} \cdot <) \rangle \psi \]
Star-free Propositional Dynamic Logic

Syntax

State formulas:
\[ \varphi ::= P \mid \varphi \lor \varphi \mid \neg \varphi \mid \langle \pi \rangle \varphi \]

Path formulas:
\[ \pi ::= \leq \mid R \mid \{ \varphi \}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^c \]

PDL_{sf}
Star-free Propositional Dynamic Logic

Syntax

State formulas:
\[ \varphi ::= P \mid \varphi \lor \varphi \mid \neg \varphi \mid \langle \pi \rangle \varphi \]

Path formulas:
\[ \pi ::= \leq \mid R \mid \{ \varphi \}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^c \]

Combines features from

- Propositional Dynamic Logic [Fisher-Ladner 1979]
- Star-free regular expressions
- The calculus of relations
Star-free Propositional Dynamic Logic

Syntax

State formulas:
\[ \varphi ::= P \mid \varphi \lor \varphi \mid \neg \varphi \mid \langle \pi \rangle \varphi \]

Path formulas:
\[ \pi ::= \leq \mid R \mid \{ \varphi \}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^c \]

Combines features from

- Propositional Dynamic Logic [Fisher-Ladner 1979]
- Star-free regular expressions
- The calculus of relations

Theorem: [Tarski-Givant 1987 (calculus of relations)]
PDL\textsubscript{sf} and FO\textsuperscript{3} are expressively equivalent
A fragment of Star-free PDL

State formulas:
\[ \phi ::= P \mid \phi \lor \phi \mid \neg \phi \mid \langle \pi \rangle \phi \]

Path formulas:
\[ \pi ::= \leq \mid R \mid \{ \phi \} \cdot \mid \pi - 1 \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^c \]

\[ PDL_{sf} \]

Lemma:
\[ \forall \pi \in PDL_{sf}, J_\pi K \text{ is interval-preserving} \]
A fragment of Star-free PDL

<table>
<thead>
<tr>
<th>State formulas:</th>
<th>Path formulas:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi ::= P \mid \varphi \lor \varphi \mid \neg \varphi \mid \langle \pi \rangle \varphi$</td>
<td>$\pi ::= \leq \mid R \mid {\varphi}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^c$</td>
</tr>
</tbody>
</table>

$\pi ::= \leq \mid R \mid \{\varphi\}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cap \pi \mid (\leq \cdot \pi \cdot \leq)^c \mid (\leq \cdot \pi \cdot \geq)^c \mid (\geq \cdot \pi \cdot \leq)^c \mid (\geq \cdot \pi \cdot \geq)^c$
A fragment of Star-free PDL

State formulas:
\[ \varphi ::= P \mid \varphi \lor \varphi \mid \neg \varphi \mid \langle \pi \rangle \varphi \]

Path formulas:
\[ \pi ::= \leq \mid R \mid \{ \varphi \}? \mid \pi^{-1} \mid \pi \cdot \pi \mid \pi \cup \pi \mid \pi^c \]

Lemma: \( \forall \pi \in \text{PDL}_{sf}^{int}, \llbracket \pi \rrbracket \) is interval-preserving
Equivalences over interval-preserving structures

FO \rightarrow PDL_{sf}^{\text{int}}

FO^3 \leftarrow PDL_{sf}

Equivalences over interval-preserving structures

\[ \text{FO} \xrightarrow{\text{def.}} \text{FO}^3 \xleftarrow{\text{def.}} \text{PDL}_{sf}^{int} \xrightarrow{\text{def.}} \text{PDL}_{sf} \]
Equivalences over interval-preserving structures

- FO
- PDL$^{\text{int}}_{\text{sf}}$
- FO$^3$
- PDL$_{\text{sf}}$

- def.
- trivial induction

We have equivalences between FO, PDL$^{\text{int}}_{\text{sf}}$, FO$^3$, and PDL$_{\text{sf}}$.
Equivalences over interval-preserving structures

- State formula $\varphi \in \text{PDL}_{sf} \leadsto \varphi^{\text{FO}}(x) \in \text{FO}$

- Path formula $\pi \in \text{PDL}_{sf} \leadsto \pi^{\text{FO}}(x, y) \in \text{FO}$
Equivalences over interval-preserving structures

- State formula $\varphi \in \text{PDL}_{sf} \leadsto \varphi^{\text{FO}}(x) \in \text{FO}$
  
  $\langle \pi \rangle \varphi \leadsto \exists y.\pi^{\text{FO}}(x, y) \land \varphi^{\text{FO}}(y)$

- Path formula $\pi \in \text{PDL}_{sf} \leadsto \pi^{\text{FO}}(x, y) \in \text{FO}$
  
  $\pi_1 \cdot \pi_2 \leadsto \exists z.\pi_1^{\text{FO}}(x, z) \land \pi_2^{\text{FO}}(z, y)$
Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\text{FO}}(x_i, x_j)$, where $\pi \in \text{PDL}^{\text{int}}_{\text{sf}}$. 
Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\text{FO}}(x_i, x_j)$, where $\pi \in \text{PDL}_{\text{sf}}^{\text{int}}$.

**Proof:** by induction on $\Phi$. 

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^\text{FO}(x_i, x_j)$, where $\pi \in \text{PDL}^\text{int}_{\text{sf}}$.

**Proof:** by induction on $\Phi$.

- Atomic formulas, disjunction: easy
Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi_{FO}^{FO}(x_i, x_j)$, where $\pi \in \text{PDL}_{sf}^{int}$.

**Proof:** by induction on $\Phi$.

- **Negation:** Express $\pi^c$ using
  
  $(\leq \cdot \pi \cdot \leq)^c, \ (\leq \cdot \pi \cdot \geq)^c, \ (\geq \cdot \pi \cdot \leq)^c, \ (\geq \cdot \pi \cdot \geq)^c$. 

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\text{FO}}(x_i, x_j)$, where $\pi \in \text{PDL}^{\text{int}}_{\text{sf}}$.

**Proof:** by induction on $\Phi$.

- **Existential quantification:** Similar to the example before.
Equivalences over interval-preserving structures

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\text{FO}}(x_i, x_j)$, where $\pi \in \text{PDL}^{\text{int}}_{\text{sf}}$.

**Proof:** by induction on $\Phi$.

- **Existential quantification:** Similar to the example before.

  $$\exists x. \bigwedge_i \pi^\text{FO}_i(x_i, x)$$
Equivalences over interval-preserving structures

Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\text{FO}}(x_i, x_j)$, where $\pi \in \text{PDL}^{\text{int}}_{\text{sf}}$.

**Proof:** by induction on $\Phi$.

- **Existential quantification:** Similar to the example before.
  $$\exists x. \bigwedge_i \pi_i^{\text{FO}}(x_i, x)$$
  intersection of $n$ intervals
Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi_{\text{FO}}^i(x_i, x_j)$, where $\pi \in \text{PDL}_{\text{sf}}^\text{int}$.

**Proof:** by induction on $\Phi$.

- **Existential quantification:** Similar to the example before.

$$\exists x. \bigwedge_i \pi_{\text{FO}}^i(x_i, x)$$

Intersection of $n$ intervals
Any FO formula $\Phi(x_1, \ldots, x_n)$ is equivalent to a finite positive boolean combination of formulas of the form $\pi^{\text{FO}}(x_i, x_j)$, where $\pi \in \text{PDL}_{\text{sf}}^{\text{int}}$.

**Proof:** by induction on $\Phi$.

- **Existential quantification:** Similar to the example before.

  $$\exists x. \bigwedge_i \pi_i^{\text{FO}}(x_i, x) \equiv \bigwedge_{i,j} (\pi_i \cdot \{\varphi\} ? \cdot \pi_j^{-1})^{\text{FO}}(x_i, x_j)$$

  - intersection of $n$ intervals
  - pairwise intersections
Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

\[
\text{FO} = \text{PDL}_{sf} = \text{FO}^3
\]
Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

\[ \text{FO} = \text{PDL}_{sf} = \text{FO}^3 \]

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...
Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

\[ \text{FO} = \text{PDL}_{sf} = \text{FO}^3 \]

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...

- Star-free PDL is a useful technical tool, but also an interesting logic on its own.
Conclusion

▶ Over linearly ordered structures with interval-preserving binary relations,

\[
\text{FO} = \text{PDL}_{sf} = \text{FO}^3
\]

▶ Covers many classical classes of structures: linear orders, real-time signals, MSCs, . . .
▶ Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:
▶ Generalizations to other types of orders (trees . . .), relations of arity $\geq 2$?
Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

\[
\text{FO} = \text{PDL}_{sf} = \text{FO}^3
\]

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, . . .
- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:
- Generalizations to other types of orders (trees . . .), relations of arity \( \geq 2 \)?
- Uniform approach for proving completeness of temporal logics?
Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

\[ FO = PDL_{sf} = FO^3 \]

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, ...
- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:

- Generalizations to ther types of orders (trees...), relations of arity \( > 2 \)?
- Uniform approach for proving completeness of temporal logics?
Conclusion

- Over linearly ordered structures with interval-preserving binary relations,

\[ \text{FO} = \text{PDL}_{sf} = \text{FO}^3 \]

- Covers many classical classes of structures: linear orders, real-time signals, MSCs, . . .

- Star-free PDL is a useful technical tool, but also an interesting logic on its own.

Further directions:

- Generalizations to other types of orders (trees...), relations of arity > 2?

- Uniform approach for proving completeness of temporal logics?

Thank you!